

5. Jacobi's Identity for the Theta Function

The "theta function" is defined by the sum

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 t), \quad t > 0,$$

and Jacobi's formula states that

$$\vartheta(t) = t^{-1/2} \vartheta(1/t).$$

The function ϑ is an important transcendental function arising in a number of very different fields, including number theory, heat flow, elliptic and automorphic functions, and statistical mechanics. The reader will find other proofs of Jacobi's identity later [see Subsection 1.8.3 and Subsection 2.7.5]. The identity is very useful for computing ϑ for small t . For example, if $t = 0.01$ and if π is known, you must take about 21 terms ($|n| \leq 10$) of the sum on the left-hand side to compute to one significant figure, while the very first term ($n = 0$) of the right-hand sum gives the correct value to over 130 significant figures!

EXERCISE 4. Convince yourself that these numbers are reasonable.

PROOF OF JACOBI'S IDENTITY. Fix $t > 0$ and look at the (periodic) function

$$f(x) = \sum_{k=-\infty}^{\infty} \exp[-(x-k)^2/2t].$$

The sum converges uniformly for $0 \leq x \leq 1$ since

$$0 \leq \exp[-(x-n)^2/2t] \leq \exp(-n^2/4t) \quad \text{for } |n| \geq 2 \quad \text{and} \quad 0 \leq x \leq 1,$$

so you can compute the Fourier coefficients of f as follows:

$$\begin{aligned} \hat{f}(n) &= \int_0^1 f e_n^* = \sum_{k=-\infty}^{\infty} \int_0^1 \exp[-(x-k)^2/2t] e^{-2\pi i n x} dx \\ &= \sum_{k=-\infty}^{\infty} \int_{-k}^{-k+1} \exp(-x^2/2t) e^{-2\pi i n x} dx \\ &= \int_{-\infty}^{\infty} \exp(-x^2/2t) e^{-2\pi i n x} dx \\ &= \sqrt{2\pi t} \exp(-2\pi^2 n^2 t). \end{aligned}$$

The last evaluation will be verified shortly. Granting this, it is clear from

the rapid decrease of \hat{f} that the formal sum $\sum \hat{f}(n) e_n$ converges uniformly to f :

$$f(x) = \sum_{n=-\infty}^{\infty} \exp[-(x-n)^2/2t] = \sqrt{2\pi t} \sum_{n=-\infty}^{\infty} \exp(-2\pi^2 n^2 t) e_n(x).$$

This identity was also known to Jacobi. It specializes to the identity for the theta function upon putting $x = 0$ and replacing t by $t/2\pi$. The actual evaluation of

$$\hat{f}(n) = \int_{-\infty}^{\infty} \exp(-x^2/2t) e^{-2\pi i n x} dx = \sqrt{2\pi t} \exp(-2\pi^2 n^2 t)$$

is carried out by a couple of elegant tricks, the first of which is due to Feller [1966, Vol. 2, p. 476]. Bring in the function

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} \exp(-x^2/2t) e^{-2\pi i \gamma x} dx$$

defined for all $\gamma \in \mathbb{R}^1$, and notice that

$$\begin{aligned} \hat{f}' &= -2\pi i \int_{-\infty}^{\infty} x \exp(-x^2/2t) e^{-2\pi i \gamma x} dx \\ &= 2\pi i t \int_{-\infty}^{\infty} [\exp(-x^2/2t)]' e^{-2\pi i \gamma x} dx \\ &= -2\pi i t \int_{-\infty}^{\infty} \exp(-x^2/2t) (e^{-2\pi i \gamma x})' dx \\ &= -4\pi^2 \gamma t \int_{-\infty}^{\infty} \exp(-x^2/2t) e^{-2\pi i \gamma x} dx \\ &= -4\pi^2 \gamma t \hat{f} \end{aligned}$$

by a self-evident partial integration. This may be solved for \hat{f} :

$$\hat{f}(\gamma) = \hat{f}(0) \exp(-2\pi^2 \gamma^2 t),$$

and to finish the proof you have only to evaluate

$$\hat{f}(0) = \int_{-\infty}^{\infty} \exp(-x^2/2t) dx = \sqrt{t} \int_{-\infty}^{\infty} \exp(-x^2/2) dx \equiv \sqrt{t} I$$

as $(2\pi t)^{1/2}$. The trick for doing that is an old standby:

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} \exp(-x^2/2) dx \int_{-\infty}^{\infty} \exp(-y^2/2) dy \\ &= \int_{\mathbb{R}^2} \exp[-(x^2+y^2)/2] dx dy \end{aligned}$$