PROBLEMS AND SOLUTIONS

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal nor posted to the Internet before the due date for solutions. Submitted solutions should arrive before June 30, 2016. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11887. *Proposed by Luke Harmon and Greg Oman, University of Colorado, Colorado Springs, CO.*
(**a**) What is the smallest cardinality of a commutative ring R, not assumed to have a multiplicative identity, which has exactly five ideals (including (0) and R)?
(**b**) Does the answer to (**a**) change if R has a multiplicative identity?

11888. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Let $a > 1$. For $x > 0$, let $L(x) = \log\log(\min\{x, e^e\})$. Define the sequence $\langle x_n \rangle$ by $x_n = aL(n_k)$ for $n_k \le n < n_{k+1}$ and $k \ge 0$, where $n_0 = 1$ and $n_k = \min\{n \ge 1 \colon L(n) \ge aL(n_{k+1})\}$ for $k \ge 1$. Prove that $\sum_{n=1}^{\infty} \frac{1}{n} e^{-(1-x)x_n} = \infty$ for all $x \ge 0$.

11889. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Let

$$a_n = \frac{(n+2)^{n+1}}{(n+1)^n},$$

and let

$$b_n(x) = n^{\sin^2 x}\left(a_{n+1}^{\cos^2 x} - a_n^{\cos^2 x}\right).$$

Find $\lim_{n\to\infty} b_n(x)$. (Here, n denotes a positive integer and x a real number.)

11890. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Find all x in $(1, \infty)$ such that

$$\left(\frac{1}{x} + \frac{x-1}{x+1}\right) + \frac{1}{3}\left(\frac{1}{x^3} + \left(\frac{x-1}{x+1}\right)^3\right) + \cdots = \frac{1}{2}\int_0^x \frac{dt}{\sqrt{1+t^2}}.$$

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11891. *Proposed by Cristian Chiser, National Elena Cuza College, Craiova, Romania.* Let M be the set of all 3×3 matrices with complex entries. Suppose $A, B \in M$ with $AB = BA$ and $\mathrm{Tr}(A^k) = \mathrm{Tr}(B^k)$ for $k \in \{1, 2, 3\}$. Suppose further that $n \geq 4$ and that $A^n - B^n$ is invertible. Prove that there exist complex numbers α, β, γ such that

$$A^{2n} + B^{2n} + A^n B^n + \alpha A^n + \beta B^n + \gamma I = 0.$$

11892. *Proposed by Francisco Perdomo and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.* Let f be a real-valued continuously differentiable function on $[a, b]$ with positive derivative on (a, b). Prove that, for all pairs (x_1, x_2) with $a \leq x_1 < x_2 \leq b$ and $f(x_1)f(x_2) > 0$, there exists $\xi \in (x_1, x_2)$ such that

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \xi - \frac{f(\xi)}{f'(\xi)}.$$

11893. *Proposed by Florin S. Pârvănescu, Slatina, Romania.* Let O be the center of a circle, let AB and CD be the perpendicular chords of this circle that do not contain O, let M be the intersection of these chords, and suppose that MA is longer than MB and MC longer than MD. Give a compass and straightedge construction of a quadrilateral inscribed in the circle with sides of lengths $|MA| + |MB|$, $|MC| + |MD|$, $|MA| - |MD|$, and $|MC| - |MD|$.

SOLUTIONS

There and Back Again

11707 [2013,469]. *Proposed by José Luis Palacios, Caracas, Venezuela.* For $N \geq 1$, consider the following random walk on the $(N + 1)$-cycle with vertices labeled $0, 1, \ldots, N$. The walk begins at vertex 0 and continues until every vertex has been visited and the walk returns to vertex 0. Prove that the expected number of visits to any vertex other than 0 is $\frac{1}{3}(2N + 1)$.

Solution by Michael Andreoli, Miami-Dade College, North Campus, Miami, FL. The (endless) random walk on the $(N + 1)$-cycle is a Markov chain with uniform stationary distribution over the $N + 1$ vertices. Let the random variable Y denote the total number of steps to visit all vertices and return to the initial vertex 0.

A *renewal* occurs at the time of the first return to the initial vertex after having visited all the other vertices. The time for each subsequent renewal has the same distribution as for the first. That is, there is a sequence of independent random variables $X_1, X_2, \ldots$, each with the same distribution as Y, with X_j being the time between the $(j - 1)$th and jth renewals. The distribution of occupancy of vertices converges exponentially rapidly to the stationary distribution, which we have noted is uniform. However, the process between the $(j - 1)$th and jth renewals is the same for all j, generating the same distribution of occupancy of vertices. Since the stationary distribution is uniform, we conclude that all vertices have the same expected number of visits during a single renewal. Letting this value be $\mathbb{E}[Z]$, where Z is the number of visits to a particular vertex, we thus obtain

$$\frac{1}{N+1} = \frac{\mathbb{E}[Z]}{\mathbb{E}[Y]}.$$

The numerator in the last fraction is the desired value; we thus complete the proof by computing $\mathbb{E}[Y] = (N + 1)(2N + 1)/3$.

To compute $\mathbb{E}(Y)$, we express Y as a sum of random variables measuring portions of the process. Let T_j denote the number of steps from the first moment when j states have been visited until the first moment when $j+1$ states have been visited; note that $T_1 = 1$. From the first moment when all states have been visited, let R denote the time for the cyclic random walk to return to 0. Thus, $Y = R + \sum_{j=1}^{N} T_j$.

To compute $\mathbb{E}(T_i)$ and $\mathbb{E}(R)$, we use two well-known results about the gambler's ruin problem, which is a random walk on $\{0, \ldots, M\}$ starting at i, ending when one endpoint is reached. The probability is i/M of ending at M and $1 - i/M$ of ending at 0, and the expected duration is $i(M-i)$ (see W. Feller, *An Introduction to Probability Theory and Its Applications* (John Wiley & Sons), Volume 1, 2nd ed. Chapter XIV).

When the random walk on the cycle first has visited j distinct vertices, the current vertex is an endpoint of a path consisting of the j visited states. The process to reach a new vertex is a gambler's ruin problem with $M = j+1$ and $i = j$; the expected duration is thus j. Therefore, $\mathbb{E}(\sum_{j=1}^{M} T_j) = N(N+1)/2$.

Let L be the vertex last to be visited by the random walk on the cycle. The process to return to 0 is another gambler's ruin problem, with $M = N+1$ and $i = L$. Thus, $\mathbb{E}[R|L=j] = j(N+1-j)$. To complete the computation of $\mathbb{E}(Y)$, we show that $\mathbb{P}[L=a] = 1/N$ for $1 \le a \le N$. Given a symmetric random walk on the integers, starting at 0, let $p_{a,N}$ denote the probability that every vertex in the interval $[a-N, a-1]$ is visited before a or $a-N-1$ is visited; note that $\mathbb{P}[L=a] = p_{a,N}$.

Let A be the event that the walk visits $a-1$ before $a-N$ and then $a-N$ before a. Let B be the event that the walk hits $a-N$ before $a-1$ and then $a-1$ before $a-N-1$. These events are disjoint, and $p_{a,N} = \mathbb{P}[A] + \mathbb{P}[B]$. Using the win probabilities in the gambler's ruin problem,

$$\mathbb{P}[L=a] = \frac{N-a}{N-1} \cdot \frac{1}{N} + \frac{a-1}{N-1} \cdot \frac{1}{N} = \frac{1}{N}.$$

Thus,

$$\mathbb{E}[R] = \sum_{j=1}^{N} \mathbb{E}[R|L=j]\mathbb{P}[L=j] = \frac{1}{N}\sum_{j=1}^{N} j(N+1-j) = \frac{(N+1)(N+2)}{6}.$$

We conclude $E[Y] = \sum_{j=1}^{N} E[T_j] + E[R] = (N+1)(2N+1)/3$, and the expected number of visits to each vertex other than 0 is $(2N+1)/3$.

Editorial comment. It bears noting that the conditional frequency of visits to a particular vertex, given that the tour duration is K, need not be the same for all vertices. It is only when all K are taken together that the visit frequencies are equal.

Also solved by M. Andreoli, M. Catalano-Johnson, J. H. Lindsey II, M. Wildon (U. K.), GCHQ Problem Solving Group (U. K.), and the proposer.

Fully Divisible Numbers

11747 [2014, 83]. *Proposed by Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI.* Determine all $n \in \mathbb{N}$ such that $\lfloor n/k \rfloor$ divides n for $1 \le k \le n$. Similarly, determine all $n \in \mathbb{N}$ such that $\lceil n/k \rceil$ divides n for $1 \le k \le n$.

Solution by Nicolás Caro, Universidade Federal de Pernambuco, Recife, Brazil. The first condition holds for 1, 2, 3, 4, 6, 8, 12, 24, the second for 1, 2, 4, 6, 12.

Let $k, n \in \mathbb{N}$ be such that $1 \le k \le n$ and $k \nmid n$. We have $n = k\lfloor n/k \rfloor + r_k = k(\lceil n/k \rceil - 1) + r_k$ with $1 \le r_k \le k-1$. If $\lfloor n/k \rfloor \mid n$, then $\lfloor n/k \rfloor \mid r_k$, so $n/k <$

$\lfloor n/k \rfloor + 1 \le r_k + 1 \le k$. Similarly, $\lceil n/k \rceil \mid n$ yields $\lceil n/k \rceil \mid (k - r_k)$, so $n/k < \lceil n/k \rceil < k$. Hence, under either requirement, $k \mid n$ for all k with $1 \le k^2 \le n$.

If $n = m^2 + j$ with $0 \le j \le 2m$ and $k \mid n$ for $1 \le k \le m$, then $m \mid j$, so $j = rm$ with $r \in \{0, 1, 2\}$. If $m \ge 2$, then $m - 1$ divides $n - (m^2 - 1)$, which equals $r(m - 1) + r + 1$. Thus, $m - 1$ divides $r + 1$, which implies $m - 1 \le r + 1 \le 3$. Now the possible values of n are 1, 2, 3 (for $m = 1$), 4, 6, 8 (for $m = 2$), 12 (for $m = 3$), and 24 (for $m = 4$). These values all satisfy the first condition, while 3, 8, and 24 are the only ones not satisfying the second condition ($\lceil 3/2 \rceil = 2$, $\lceil 8/3 \rceil = 3$, $\lceil 24/5 \rceil = 5$).

Also solved by C. Blatter (Switzerland), B. S. Burdick, M. A. Carlton, R. Chapman (U. K.), C. T. R. Conley, N. Curwen (U. K.), D. Fleischman, S. M. Gagola Jr., O. Geupel (Germany), J. T. Goodman, E. A. Herman, Y. J. Ionin, J. C. Kieffer, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Mabry, R. Martin (Germany), V. Pambuccian, R. E. Prather, N. C. Singer, J. H. Smith, R. Stong, T. Viteam (South Africa), E. A. Weinstein, M. Wildon (U. K.), T. Woodcock, CMC 328, GCHQ Problem Solving Group (U. K.), and the proposer.

Maximize a Moment of Inertia

11760 [2013, 171]. *Proposed by Stefano Siboni, University of Trento, Trento, Italy.* Let D be the closure of a simply connected, bounded open subset of $\mathbb{R}^2$. Let W be the subset of $[0, 1]^n$ consisting of all points $(w_1, \ldots, w_n)$ such that $w_1 + \cdots + w_n = 1$. Let g be a point in D, and let n be an integer, $n > 1$. With $p = (p_1, \ldots, p_n) \in D^n$, let M be the function from $D^n \times W$ to $\mathbb{R}$ given by

$$M(p, w) = \sum_{k=1}^{n} w_k \|p_k - g\|^2,$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$.
(**a**) Show that if $M(p, w)$ is maximized at (p', w'), then all entries of p' lie on the boundary of D.
(**b**) Restricting now to the case in which $n = 2$ and the boundary of D is an ellipse, let $((p_1', p_2'), (w_1', w_2'))$ be a point at which $M((p_1, p_2), (w_1, w_2))$ is maximized. Show that p_1' and p_2' lie opposite each other on the major axis of the ellipse.
(**c**) Show that if D is a disk of radius r about the origin, then the maximum value of M is $r^2 - \|g\|^2$.

Editorial comment. The published problem was garbled when the editors attempted to simplify it. First, the function M is to be maximized over the set K_g of all (p, w) such that $\sum_{k=1}^{n} w_k p_k = g$ so that M is the moment of inertia of n masses w_k distributed at the n points p_k with center of mass g. Second, the conclusion of (**b**) should be: p_1' and p_2' are opposite ends of the chord through g parallel to the major axis.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. (**a**) Applying a translation if necessary, we may assume that $g = 0$. Suppose (for purposes of contradiction) that not all the entries $(p_1', \ldots, p_n')$ of p' lie on the boundary of D. By symmetry, assume p_n' lies in the interior so that $(1 + \varepsilon)p_n' \in D$ for some small $\varepsilon > 0$. Let $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_n)$ and $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_n)$ be defined by

$$\tilde{p}_n = (1 + \varepsilon)p_n', \qquad \tilde{p}_k = p_k' \text{ for } 1 \le k < n,$$

$$\tilde{w}_m = \frac{w_n'}{1 + \varepsilon(1 - w_n')}, \qquad \tilde{w}_k = \frac{(1 + \varepsilon)w_k'}{1 + \varepsilon(1 - w_n')} \text{ for } 1 \le k < n.$$

Note that $\tilde{w} \in W$ and $(\tilde{p}, \tilde{w}) \in K_g = K_0$. Moreover,

$$\begin{aligned}
M(\tilde{p}, \tilde{w}) &= \frac{1+\varepsilon}{1+\varepsilon(1-w'_n)} \left((1+\varepsilon) w'_n \|p'_n\|^2 + \sum_{k=1}^{n-1} w'_k \|p'_k\|^2 \right) \\
&= \frac{1+\varepsilon}{1+\varepsilon(1-w'_n)} \left(\varepsilon w'_n \|p'_n\|^2 + M(p', w') \right) \\
&= M(p', w') + \frac{\varepsilon w'_n}{1+\varepsilon(1-w'_n)} \left((1+\varepsilon)\|p'\|^2 + M(p', w') \right) \\
&> M(p', w').
\end{aligned}$$

This contradiction proves that p'_n must belong to the boundary of D, which proves (**a**).

(**b**) Let the boundary of D be the ellipse $\mathcal{E}$ with equation $x^2/a^2 + y^2/b^2 = 1$, $a > b$, and suppose that $g = (u, v)$ is a point in the interior of D. We may consider $\mathcal{E}$ as the projection on the xy-plane of a circle in three-dimensional xyz-space. Precisely, let $\mathcal{C}$ be the circle of radius a obtained as the intersection $\mathcal{S} \cap \mathcal{P}$ of the sphere $\mathcal{S}$ of radius a centered at the origin and the plane $\mathcal{P}$ with equation $cz = by$ (where $c = \sqrt{a^2 - b^2}$). The projection π given by $\pi(x, y, z) = (x, y)$ defines a linear isomorphism between $\mathcal{P}$ and the xy-plane, with $\pi(x, y, by/c) = (x, y)$. We have $\pi(\mathcal{C}) = \mathcal{E}$.

Consider $(p, w) \in K_g$ with $p = (p_1, p_2)$ and $w = (w_1, w_2)$. Let $\tilde{p}_1, \tilde{p}_2, \tilde{g} \in \mathcal{P}$ such that $\pi(\tilde{p}_1) = p_1$, $\pi(\tilde{p}_2) = p_2$, and $\pi(\tilde{g}) = g$. Then $\tilde{g} = w_1 \tilde{p}_1 + w_2 \tilde{p}_2$. Writing M for $M(p, w)$, we compute

$$\begin{aligned}
M &= w_1 \|p_1 - g\|^2 + w_2 \|p_2 - g\|^2 = w_1 \|\pi(\tilde{p}_1 - \tilde{g})\|^2 + w_2 \|\pi(\tilde{p}_2 - \tilde{g})\|^2 \\
&\le w_1 \|\tilde{p}_1 - \tilde{g}\|^2 + w_2 \|\tilde{p}_2 - \tilde{g}\|^2 = w_1 \|\tilde{p}_1\|^2 + w_2 \|\tilde{p}_2\|^2 - \|\tilde{g}\|^2 \le a^2 - \|\tilde{g}\|^2
\end{aligned} \tag{1}$$

with equality if and only if

$$\|\tilde{p}_1\| = \|\tilde{p}_2\| = a, \quad \|p_1 - g\| = \|\pi(\tilde{p}_1 - \tilde{g})\|, \quad \text{and} \quad \|p_2 - g\| = \|\pi(\tilde{p}_2 - \tilde{g})\|. \tag{2}$$

So equality implies that p_1, p_2 lie on $\mathcal{E}$, and the straight line $\widetilde{\mathcal{L}}$ passing through $\tilde{p}_1$, $\tilde{p}_2$ (and $\tilde{g}$) is parallel to the xy-plane. But $\widetilde{\mathcal{L}}$ is contained in $\mathcal{P}$, so $\widetilde{\mathcal{L}}$ is parallel to the x-axis. Consequently, the straight line $\mathcal{L}$ through p_1, p_2 (and g) is also parallel to the x-axis. Conversely, if p_1 and p_2 are the points of intersection of the line through g parallel to the major axis of $\mathcal{E}$, then conditions (2) are satisfied, and M attains its maximum value given by (1),

$$a^2 - \|g\|^2 = a^2 - \left(u^2 + v^2 + \frac{b^2 v^2}{c^2} \right) = a^2 \left(1 - \frac{u^2}{a^2} - \frac{v^2}{b^2} \right)$$

at $((p_1, p_2), w)$ for appropriate w. This concludes the proof of (**b**).

(**c**) Suppose D is a disk of radius r centered at the origin and that M is maximized at $(p', w') \in K_g$. Then

$$\begin{aligned}
M(p', w') &= \sum_{k=1}^{n} w'_k \|p'_k - g\|^2 = \sum_{k=1}^{n} w'_k \|p'_k\|^2 - 2 \left\langle \sum_{k=1}^{n} w'_k p'_k, g \right\rangle + \|g\|^2 \\
&= \sum_{k=1}^{n} w'_k r^2 - \|g\|^2 = r^2 - \|g\|^2.
\end{aligned}$$

We used from (**a**) the fact that $w'_k \|p'_k\|^2 = w'_k r^2$ for every $k \in \{1, \ldots, n\}$.

Also solved by R. Chapman (U. K.), W. J. Cowieson, B. Karaivanov, R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

A Product Bessel Integral

11764 [2013, 267]. *Proposed by George Lamb, Tucson, AZ.* Let

$$f(a, b, \phi, \theta) = \sin(2\theta)\sin(2\phi) J_0(a\cos\theta\sin\phi) J_0(b\sin\theta\cos\phi),$$

where J_0 is the Bessel function of order 0. Show that

$$\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} f(a, b, \phi, \theta)\, d\phi\, d\theta = 4\frac{J_0(a) - J_0(b)}{b^2 - a^2}.$$

Solution by Eugene Herman, Grinnell College, Grinnell, IA. The Bessel function $J_0(x)$ has an absolutely convergent series expansion

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (n!)^2} x^{2n}.$$

It follows that

$$J_0(a\cos\theta\sin\phi) J_0(b\sin\theta\cos\phi)$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{(m!)^2(n!)^2 4^{m+n}} a^{2m} b^{2n} \cos^{2m}\theta \sin^{2n}\theta \cos^{2n}\phi \sin^{2m}\phi.$$

The integrals over the angular variables decouple and the individual integrals may be computed in closed form:

$$\int_0^{\pi/2} \sin(2\phi)\sin^{2m}\phi\cos^{2n}\phi\, d\phi = 2\int_0^{\pi/2} \sin^{2m+1}\phi\cos^{2n+1}\phi\, d\phi$$
$$= 2\int_0^1 u^{2m+1}(1-u^2)^n\, du = 2\sum_{k=0}^{n}\binom{n}{k}(-1)^k \int_0^1 u^{2m+1+2k}\, du$$
$$= \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{n+1+k} = \frac{1}{(m+n+1)\binom{m+n}{n}}.$$

By symmetry, the integral over θ can be calculated in the same way. Therefore, we have (summing the finite geometric series in going from the second to the third line)

$$\int_0^{\pi/2}\int_0^{\pi/2} f(a, b, \phi, \theta)\, d\phi\, d\theta = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{(m!)^2(n!)^2 4^{m+n}} \frac{a^{2m}b^{2n}}{(m+n+1)^2\binom{m+n}{n}^2}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{((m+n+1)!)^2 4^{m+n}} a^{2m}b^{2n} = \sum_{p=0}^{\infty} \frac{(-1)^p}{((p+1)!)^2 4^p} b^{2p} \sum_{m=0}^{p}\left(\frac{a}{b}\right)^{2m}$$
$$= \sum_{p=0}^{\infty} \frac{(-1)^p}{((p+1)!)^2 4^p} \frac{b^{2(p+1)} - a^{2(p+1)}}{b^2 - a^2} = \frac{4}{b^2 - a^2}\sum_{p=1}^{\infty} \frac{(-1)^p}{(p!)^2 4^p}(a^{2p} - b^{2p})$$
$$= 4\frac{J_0(a) - J_0(b)}{b^2 - a^2}.$$

Also solved by R. Bagby, P. Bracken, R. Chapman (U. K.), H. Chen, M. L. Glasser, J. A. Grzesik, O. Kouba (Syria), K. D. Lathrop, O. P. Lossers (Netherlands), L. Mannion, R. Stong, E. I. Verriest, GCHQ Problem Solving Group (U. K.), and the proposer.

A Product Inequality

11766 [2014, 267]. *Proposed by Ming Fang and Cherng-tiao Perng, Norfolk State University, Norfolk, VA.* Let m and n be positive integers. For $(i, j) \in [1, n] \times [1, m]$, let x_i, α_i, y_j, and β_j be real numbers with $0 \le x_i < \alpha_i < 1/2$ and $0 \le y_j < \beta_j < 1/2$.

Given that $\prod_{i=1}^{n}(1+x_i)\prod_{j=1}^{m}(1-y_j) = \prod_{i=1}^{n}(1+\alpha_i)\prod_{j=1}^{m}(1-\beta_j)$, show that

$$\prod_{i=1}^{n}(1+2\alpha_i)\prod_{j=1}^{m}(1-2\beta_j) < \prod_{i=1}^{n}(1+2x_i)\prod_{j=1}^{m}(1-2y_j).$$

Solution by Robin Chapman, University of Exeter, Exeter, U. K. Let $f(t) = (1+2t)/(1+t)^2$ for $t > -1$. Then $f'(t) = -2t/(1+t)^3$. For $t > -1$, $f'(t)$ has the opposite sign to t. Therefore, on the interval $(-1/2, 0)$, f' is positive and f is strictly increasing, while on the interval $(0, 1/2)$, f' is negative and f is strictly decreasing. Therefore, $f(x_i) > f(\alpha_i)$ and $f(-y_j) > f(-\beta_j)$. As all these numbers are positive,

$$\prod_{i=1}^{n} f(x_i)\prod_{j=1}^{m} f(-y_j) > \prod_{i=1}^{n} f(\alpha_i)\prod_{j=1}^{m} f(-\beta_j),$$

that is,

$$\prod_{i=1}^{n}\frac{1+2x_i}{(1+x_i)^2}\prod_{j=1}^{m}\frac{1-2y_j}{(1-y_i)^2} > \prod_{i=1}^{n}\frac{1+2\alpha_i}{(1+\alpha_i)^2}\prod_{j=1}^{m}\frac{1-2\beta_j}{(1-\beta_j)^2}.$$

Multiplying by the square of the given identity,

$$\prod_{i=1}^{n}(1+x_i)^2\prod_{j=1}^{m}(1-y_i)^2 = \prod_{i=1}^{n}(1+\alpha_i)^2\prod_{j=1}^{m}(1-\beta_j)^2,$$

we obtain

$$\prod_{i=1}^{n}(1+2x_i)\prod_{j=1}^{m}(1-2y_j) > \prod_{i=1}^{n}(1+2\alpha_i)\prod_{j=1}^{m}(1-2\beta_j).$$

Also solved by O. Geupel (Germany), Y. J. Ionin, B. Karaivanov, O. P. Lossers (Netherlands), R. Martin (Germany), P. Perfetti (Italy), R. Stong, GCHQ Problem Solving Group (U. K.), and the proposers.

A Limit with Integrals

11768 [2014, 365]. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let f be a bounded continuous function mapping $[0, \infty)$ to itself. Find

$$\lim_{n\to\infty} n\left(\sqrt[n]{\int_0^\infty f^{n+1}(x)e^{-x}\,dx} - \sqrt[n]{\int_0^\infty f^{n}(x)e^{-x}\,dx}\right).$$

Solution by Richard Bagby, New Mexico State University, Las Cruces, NM. We show that the limit is $M \log M$, when $M := \sup f(x) > 0$. (In the case $M = 0$, the limit is of course 0.) Writing

$$\lambda_n := \sqrt[n]{\int_0^\infty f(x)^n e^{-x}\, dx},$$

the limit we seek is $\lim_{n\to\infty} n\left(\lambda_{n+1}^{(n+1)/n} - \lambda_n\right)$. We first prove that $\{\lambda_n\}_{n=1}^\infty$ is a nondecreasing sequence with limit M.

Trivially, $\lambda_n \le M$ for all n. Using Hölder's inequality with exponents $(n+1)/n$ and $(n+1)$, we see that

$$\lambda_n^n = \int_0^\infty f(x)^n e^{-x} dx \le \left(\int_0^\infty f(x)^{n+1} e^{-x}\right)^{n/(n+1)} \left(\int_0^\infty e^{-x} dx\right)^{1/(n+1)} = \lambda_{n+1}^n.$$

Thus, $\lambda_n \le \lambda_{n+1}$, so $\lim_{n\to\infty} \lambda_n$ exists and is at most M. On the other hand, for every μ with $0 < \mu < M$, note that

$$\lambda_n^n \ge \mu^n \int_{\{x:\, f(x)>\mu\}} e^{-x} dx = \mu^n \theta$$

for some $\theta \in (0, 1]$ since the integration is over a nonempty open set. Thus,

$$\lim_{n\to\infty} \lambda_n \ge \mu \lim_{n\to\infty} \theta^{1/n} = \mu.$$

Also note

$$\lambda_{n+1}^{n+1} = \int_0^\infty f(x)^{n+1} e^{-x} dx \le M \int_0^\infty f(x)^n e^{-x} dx = M\lambda_n^n.$$

Therefore,

$$n\left(\lambda_n^{(n+1)/n} - \lambda_n\right) \le n\left(\lambda_{n+1}^{(n+1)/n} - \lambda_n\right) \le n(M^{1/n} - 1)\lambda_n.$$

Since

$$n\left(\lambda_n^{(n+1)/n} - \lambda_n\right) = \lambda_n \log \lambda_n \left(\frac{e^{(\log \lambda_n)/n} - 1}{(\log \lambda_n)/n}\right) \ge \lambda_n \log \lambda_n \to M \log M$$

and

$$n\left(M^{1/n} - 1\right) = (\log M)\left(\frac{e^{(\log M)/n} - 1}{(\log M)/n}\right)\lambda_n \to (\log M)(1)M,$$

we see that $M \log M$ is also the limit of the original sequence.

Editorial comment. Perrson and Sundquist and Stenger note that the change of variables $t = e^{-x}$, $g(t) = f(-\log t)$ transforms the integrals into $\int_0^1 g^n(t)\, dt$ and $\int_0^1 g^{n+1}(t)\, dt$. Persson and Sundquist (along with Vowe) then note that this problem is solved in the same way as Problem 1910 in *Mathematics Magazine* **85** (2012), p. 385, by the same proposer. Geupel, Stenger, and the New York Math Circle note that similar problems can be found in the proposer's book, *Limits, Series, and Fractional Part Integrals.*

Also solved by M. Bataille (France), P. Bracken, H. Chen, P. P. Dályay (Hungary), P. J. Fitzsimmons, O. Geupel (Germany), N. Grivaux (France), J. A. Grzesik, E. J. Ionascu, O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), P. Perfetti (Italy), T. Persson & M. P. Sundqvist (Sweden), Á. Plaza (Spain), A. Stenger, R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), GCHQ Problem Solving Group (U. K.), New York Math Circle, and the proposer.