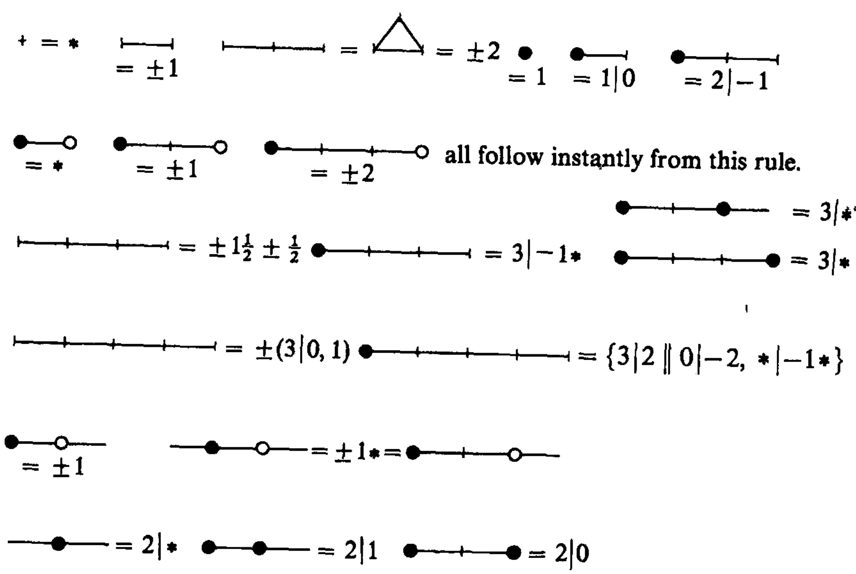
A SHORT SNORT DICTIONARY

It is much harder to do justice to SNORT positions, although I feel that in fact SNORT has a much richer theory than COL. There are some inequality and equality rules like those for COL, but since they are less frequently applicable we do not give many. Perhaps the most valuable rule is that if you can move in a node that is adjacent to every node not your own colour, you should do so. Our abbreviated notation is explained in Chapters 10 and 15.



Perhaps it is fortunate that positions in SNORT games tend to break up rapidly, and that we can delete edges joining two nodes with the same tint, so that in practice we need only tabulate the values of small positions. Highly connected positions succumb easily to the above rule, so that in fact it is long chains that are hardest to analyse. The reader should have little difficulty in finding the best move in actual play, even for quite large positions.

Larger COL and SNORT dictionaries will be found in *Winning Ways*.

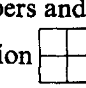
CHAPTER 9

On Games and Numbers

And now there came both mist and snow,
And it grew wondrous cold:
And ice, mast-high, came floating by,
As green as emerald.

Samuel Taylor Coleridge,
The Ancient Mariner

We know that not all games are numbers, and that for example the game $* = \{0\,|\,0\}$ is not a number, since it is confused with 0. But since for every positive number x, we have $-x < * < x$, and since we have the equality $* + * = 0$, we can confidently handle all games whose values can be expressed as sums of numbers and $*$.

But the position  in dominoes, which is equivalent to the position +—+ in SNORT, has the rather worse value $\{1\,|\,-1\}$. This game G is strictly less than all numbers greater than 1, strictly greater than all numbers less than -1, and confused with all numbers between -1 and 1 inclusive. But fortunately once again, we have $G + G = 0$, so that at least the situation does not get more complicated when we consider multiples of G.

Now in general we can get a lot of information about an arbitrary game G by comparing it with all numbers. The game G will define two "Dedekind sections" in the Class of all numbers (the Left and Right values), and any number between these two sections will be confused with G, while numbers above the greatest or below the least will be comparable with G in the appropriate sense.

This information tells us between which limits G lies, but there is also a *mean value* of G, which tells us where its centre of mass lies. We shall give algorithms for computing the Left, Right, and mean values in this Chapter.

Unfortunately, there is a large part of the argument that is inapplicable to the general infinite game. We adopt the convention of considering only *short* games in detail from now on, until Chapter 16, when we consider the differences between short games and long ones. A *short* game is one which has only finitely many positions in all. But we always explicitly add this adjective to the hypotheses of any theorem which needs it, and often add comments on general games later.

THEOREM 55. (The Archimedean principle.) *For any short game G, there is some integer n with $-n < G < n$.*

[For general G, there is some ordinal α with $-\alpha < G < \alpha$.]

Proof. Take n greater than the total number of positions in G, and consider playing in $G + n$. Left can win this by just decreasing n by 1 each time he moves, waiting for Right to run himself down in G. Since $G + n > 0$, we have $G > -n$, and similarly $G < n$.

[In general we give an inductive proof, taking for α the least ordinal greater than all α_L, α_R.]

THE LEFT AND RIGHT VALUES

We need to know which numbers x have $x \geqslant G$, and which y have $y \leqslant G$. These conditions define two *Dedekind sections* in the Class of all numbers, called the *Left section* $L(G)$ and the *Right section* $R(G)$, as follows.

A number x is put into the right-hand part of $L(G)$ iff $x \geqslant G$, and so in the left-hand part if $x \lhd\!| \, G$, while y is put into the left part of $R(G)$ if $y \leqslant G$, the right part if $y \,|\!\rhd G$.

In particular, if z is any number, $L(z)$ has for its left part all numbers strictly less than z, z and greater numbers forming its right part, while $R(z)$ has z and smaller numbers to its left, greater numbers to its right.

So $L(z)$ and $R(z)$ are the sections just to the left and right of z, respectively. For a more general game G, if $L(G)$ is one of the two sections $L(x)$, $R(x)$ for some number x, we call x the *Left value* $L_0(G)$ of G, while y is called the *Right value* $R_0(G)$ if $R(G) = L(y)$ or $R(y)$.

We introduce the obvious order on sections ($S < T$ if some number is to the right of S and the left of T), so that $L(z) < R(z)$ for each number z. But for other games, the inequality goes the other way, for if $L(G) < x < R(G)$, we have $x \leqslant G \leqslant x$, and so $G = x$. How do we compute these sections, in general?

THEOREM 56. *We have* $L(G) = \max\limits_{G^L} R(G^L) = L$, *say,*

$$\text{and } R(G) = \min_{G^R} L(G^R) = R, \text{ say}$$

unless $L < R$, when G is a number, namely the simplest number x satisfying $L < x < R$, when we have $L(G) = L(x)$, $R(G) = R(x)$.

[For general G, we must replace max and min by sup and inf.]

Proof. We tackle the case $L < R$ first. If x is the simplest number between, then

$$x^L < L < x < R < x^R,$$

so the moves from $G - x$ to $G - x^L$, $G - x^R$ are no good. But neither, in view of the definition of L and R, are those to $G^L - x$ and $G^R - x$, so that $G - x$, having no good move for either player, is a zero game.

In the case that $L \geqslant R$, the moves to $G^L - x$, $G^R - x$ are bad for the same reason, if $x > L$, $x < R$, respectively. So we need only consider, if $x > L$, moves to $G - x^R$, and if $x < R$, moves to $G - x^L$. But these fail, since we have $x^R > x > L$ in the first case, and $x^L < x < R$ in the second.

STOPPING POSITIONS

When the value of a position *is* a number, neither player will wish to move in it, for any move by Left will decrease the value, and any move by Right increase it. We can be kind to the players and agree to *stop* the game (possibly before its real end) as soon as the value becomes a number, and score positive values in favour of Left, negative ones in favour of Right. So we shall call positions of G which are equivalent to numbers the *stopping positions* of G.

Now Left will naturally prefer to arrange that when the game stops in this sense, its value will be as large as possible, while Right will prefer to make it small. If they play in this way, the value of the game when its stops will be a perfectly definite number which depends only on who starts. Moreover, each player will prefer that when the game stops it is his opponent who is about to move (and so do himself some harm).

Now we can describe the situation by saying that if Left starts, the game will end at some number x, with some player P (Left or Right) about to play, by the equality $L(G) = P(x)$, and the corresponding assertion that if Right starts the game will end at a number y with Q about to play, by the equality $R(G) = Q(y)$. This is because Theorem 56 tells us that the Left and Right sections of G are computed exactly as we should compute the numbers x and y, and locate the players P and Q.

Summary. We can determine exactly what are the order relations between a game G and *all* numbers by simply playing G intelligently until it stops and then noting the value and who is about to play.

Examples

The game $\{5 \mid 4, 7\}$. In this game, if Left starts, the game will end at 5, with Right to play, and so $L(G) = R(5)$, the section "just to the right" of 5.

If Right starts, the game ends with Left to play, at the number 4, if Right has any sense, and so $R(G) = L(4)$, just to the left of 4. We conclude that G is *strictly less* than all numbers greater than 5, *strictly greater* than all numbers less than 4, and *confused* with all numbers between 4 and 5 inclusive.

The game $\{9 \mid \{7 \mid 2\}\}$. Here $L(G) = R(9)$, the argument being as before, but we have $R(G) = R(7)$, for if Right starts, moving to $\{7 \mid 2\}$, Left continues the game for one more move, before it stops at value 7 with Right to play. So the game is *less* than numbers greater than 9, *greater* than numbers less than *or equal to* 7, and *confused* with numbers between 7 (exclusive) and 9 (inclusive).

The game $\{\{3 \mid 0\} \mid \{\frac{1}{2} \mid 9\}\}$. Here if Left starts we arrive at $L(0)$, while if Right starts we stop at $R(\frac{1}{2})$. But these are *not* the Left and Right sections of G, for we have $R(\frac{1}{2}) > L(0)$. So in this case, G is a number, namely the simplest number x satisfying $L(0) < x < R(\frac{1}{2})$, namely 0 itself. So in fact we have $L(G) = L(0)$, $R(G) = R(0)$, $G = 0$.

If we had replaced the position 0 here by $\frac{1}{4}$, the answer would have been $\frac{1}{2}$; if by -1, the answer would still have been 0; and if by $+1$, we would no longer have had a number, and $L(G) = L(1)$, $R(G) = R(\frac{1}{2})$.

Moral. When computing Left and Right values, look out for the inequality $L < R$ between Left and Right sections.

The games $*$ *and* $\uparrow$. Since $* = \{0 \mid 0\}$, we have $L(*) = R(0)$, $R(*) = L(0)$. We need not beware, since L is safely greater than R, and we conclude that $*$ is greater than all negative numbers, less than all positive numbers, but confused with 0. Again, since $\uparrow = \{0 \mid \{0 \mid 0\}\}$, we find $L(\uparrow) = R(0)$, $R(\uparrow) = R(0)$, and so $\uparrow$ is strictly positive (as we knew) but strictly less than all positive numbers. (Note that for $\uparrow$, we had $L = R$, so *almost* had to beware, etc. But not quite!)

So these games are infinitesimal in a totally new sense, for we have, for instance,

$$0 < \uparrow < \frac{1}{\omega}, \quad 0 < \uparrow < \frac{1}{\varepsilon_0}, \quad 0 < \uparrow < \frac{1}{2^{\aleph_0}}, \ldots$$

($2^{\aleph_0}$ being identified with the smallest ordinal having that cardinal), and so on. (Informally, $0 < \uparrow < 1/\mathbf{On}$.) Rather than invent some long adjective to qualify the word infinitesimal in this sense, we simply call such games *small*. So a *small* game is any game G for which we have $-x < G < x$ for every possible positive *number* x. Some small games (like $\uparrow$) are positive, others (like $\downarrow$) negative, and still others (like $*$) are fuzzy, while of course zero is itself a small game. So the small World is indeed a microcosm of the larger one.

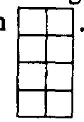
THE ALL SMALL GAMES

We call a game *all small* if all its positions are small games.

THEOREM 57. *G is all small if and only if every stopping position of G is zero.*

Proof. If some position of G were a non-zero number, it would be a non-small position of G. So we need only prove that if all the stopping positions are zero, then so are the Left and Right values. This follows immediately from Theorem 56.

Note. There are positive games smaller than all positive all small games. One such is the value $\{0 \mid \{0 \mid -2\}\}$ of the domino position



. The multiples of $\uparrow$ are among the largest of all small games.

THE MEAN VALUE THEOREM

We shall prove that for every short game G there is a real number m, called the *mean value* $m(G)$, such that for every finite n, the game nG is "nearly equal" to nm. This result, for a slightly different class of games, was first conjectured by J. Milnor, and first proved for that class by O. Hanner. A simplified proof, for the Class of games considered here, was given by Elwyn Berlekamp. All these proofs depend on a fairly complicated analysis that yields a strategy for playing nG so as to ensure a stopping value near the desired mean value nm.

The first proof given here is the remarkable "1-line" proof found by Simon Norton, which proves the existence of the mean value and finds good bounds for nG, but which does not enable us to compute this value! Then we shall give another proof, found by Norton and the author jointly, which gives us an easy algorithm for computing the mean value and much other information. This new proof formalises and simplifies an idea whose germ is found in the papers of Milnor and Hanner but which was discovered only after a completely independent analysis.

We start with some obvious inequalities about the Left and Right values $L_0(G)$, $R_0(G)$. Recall that these are the numbers next to the sections $L(G)$ and $R(G)$.

THEOREM 58. *We have*

$$R_0(G) + R_0(H) \leqslant R_0(G + H) \leqslant R_0(G) + L_0(H) \leqslant L_0(G + H) \leqslant L_0(G) + L_0(H).$$

Proof. These are obvious in terms of strategies. Thus Left, playing second in $G + H$, can guarantee a stopping value of at least $R_0(G) + R_0(H)$ by replying always in the component Right moves in, and following in that component his strategy yielding its Right value. The others can be proved similarly, but are in fact equivalent to this one. For instance

$$R_0(G) = R_0(G + H - H) \geqslant R_0(G + H) + R_0(-H) = R_0(G + H) - L_0(H).$$

THEOREM 59. *(The mean value theorem.) For every short game G there is a number $m(G)$ and a number t (both real) such that*

$$nm(G) - t \leqslant nG \leqslant nm(G) + t$$

for all finite integers n.

Proof. After the previous theorem, it will suffice to prove that $L_0(nG)$ and $R_0(nG)$ have a difference bounded independently of the number n, for then $(1/n)R_0(nG)$ and $(1/n)L_0(nG)$ must converge to a common value $m(G)$, since we have the inequalities

$$R_0(G) \leqslant \frac{1}{n} R_0(nG) \leqslant \frac{1}{n} L_0(nG) \leqslant L_0(G).$$

But we have

$$R(nG) \leqslant L(nG) = R((n-1)G + G^L) \leqslant R(nG) + L(G - G^L)$$

for the G^L for which the max in Theorem 56 is attained.

Note. The proof shows also that the number t is bounded by max $L_0(G - G^L)$, and similarly, bounded by max $L_0(G^R - G)$. These inequalities will be improved later.

THE TEMPERATURE THEORY

We can regard the game G as vibrating between its Right and Left values in such a way that on average its centre of mass is at $m(G)$. So in order to compute $m(G)$ we must find some way of cooling it down so as to quench these vibrations, and perhaps if we cool it sufficiently far, it will cease to vibrate at all, and *freeze* at $m(G)$.

Now the heat in a game comes largely from the excitement of playing it—if there are positions in G from which each player can gain tremendously by making a suitable move, then G will naturally be very heated! So for instance the game $\{1000 \mid -1000\}$ is a very hot position, for although its mean value is zero, the player who moves first in it stands to gain 1000. On the natural scale, the temperature of this game is 1000°.

On this theory, we should be able to cool G through a temperature of t° by making it just that much less exciting to move in each position of G that has not already stopped. So we shall define a new game G_t (*G cooled by t*) by charging each player a fee of t every time he makes a move, until the value becomes a number. A formal definition is complicated slightly by the need to detect when this has taken place.

Definition. If G is a short game, and t a real number $\geqslant 0$, then we define the *cooled game G_t* by the formula

$$G_t = \{G^L{}_t - t \mid G^R{}_t + t\},$$

unless possibly this formula defines a number (which it will for all sufficiently large t). For the smallest values of t for which this happens, the number turns out to be constant (that is, independent of t), and we define G_t to be this constant number for all larger t.

[The reader will see that our definition of G_t contains an assertion, and so does not really count as a definition until this assertion is verified to hold for all short G. The reason the theory does not work for general games G is that this assertion fails to hold for certain long games G.]

To see how the definition works, we treat the case $G = \{4 \mid 1\}$, supposing it already established that $4_t = 4$, $1_t = 1$ for all t. Then our formula gives $G_t = \{4 - t \mid 1 + t\} = G(t)$ unless perhaps when $G(t)$ is a number, when...? When is $G(t)$ a number? Obviously when t exceeds $1\frac{1}{2}$. What number is $G(t)$? The answer to this question depends on t, and in fact we have

$$\begin{aligned} G(t) = 2\tfrac{1}{2} &\text{ for } 1\tfrac{1}{2} < t \leqslant 2 \\ 2 &\text{ for } 2 < t \leqslant 3 \\ 1 &\text{ for } 3 < t \leqslant 4 \\ 0 &\text{ for } 4 < t. \end{aligned}$$

So as the definition asserts, $G(t)$ is a constant number $(2\frac{1}{2})$ for all the smallest numbers t for which it is a number (namely the numbers t with $1\frac{1}{2} < t \leqslant 2$), and so we have $G_t = \{4 - t \mid 1 + t\}$ for $0 \leqslant t \leqslant 1\frac{1}{2}$, and $G_t = 2\frac{1}{2}$ for *all* larger t.

We define the sections $L_t(G)$ and $R_t(G)$ to be $L(G_t)$ and $R(G_t)$.

THEOREM 60. *For all short games G and real numbers $t \geqslant 0$, we have*

$$L_t(G) = \max R_t(G^L) - t = L_t, \text{ say,}$$

and

$$R_t(G) = \min L_t(G^R) + t = R_t, \text{ say,}$$

unless possibly $L_t < R_t$. In this latter case, G_t is a number x, namely the simplest *number between L_u and R_u for all small enough u with $L_u < R_u$, and we then have $L_t(G) = L(x)$, $R_t(G) = R(x)$.*

Proof. This follows immediately on applying Theorem 56 to G_t. For the moment, we are continuing to suppose that G_t is well-defined.

THE THERMOGRAPH OF *G*

We find it convenient to describe the various numbers associated with G on a diagram. The Left options of the G with which we are concerned will usually be greater than the Right ones, so we shall reverse the normal convention and put positive values on the left, and negative ones on the right. (This happy convention has various other advantages which will appear gradually.) The temperature scale is vertical, and at height t we indicate the Left and Right values of G_t, which define the Left and Right boundaries of the *thermograph* of G. (We are indebted to Elwyn Berlekamp for this snappy substitute for our own phrase "thermal diagram".)

As our example, we take the game $G = \{\{7\,|\,5\}\,|\,\{4\,|\,1\}\}$. The calculation of the thermal properties of this game is illustrated in Fig. 15, the game itself being drawn below its thermograph. Since the games 7, 5, 4 and 1 are already numbers, they remain constant when cooled by arbitrary t, so that their thermographs are vertical lines above the appropriate numbers.

Now the Left boundary $L_t(H)$ for the game $H = \{7\,|\,5\}$ is obtained, at any rate until H_t becomes a number, by subtracting t from the Right boundary of the game 7. Since this is vertical, and subtraction corresponds to moving

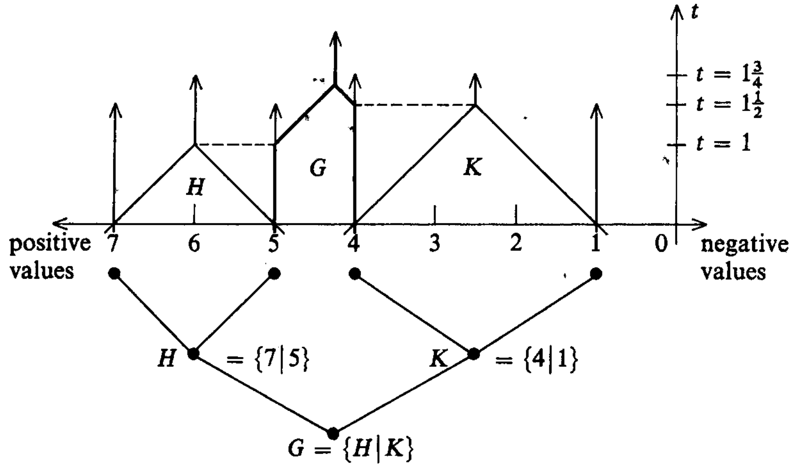


FIG. 15. Computing thermographs.

right in the diagram, this gives a line starting at 7 and moving diagonally up and right. Similarly the Left boundary is a line starting at 5 and initially moving diagonally up and left. But since these lines meet at a height of 1 above the number 6, H_t will be the constant number 6 for all t larger than 1, and the Left and Right boundaries will be vertical above this point.

So the thermograph of H is *the pyramid* ╱7, 5╲—that is to say, an isosceles right-angled triangle with hypotenuse on this interval, except that, like all thermal diagrams, it has a mast on top. The Right boundary of this diagram consists of the right side of the triangle together with the mast.

In a similar way, the game $K = \{4\,|\,1\}$ yields the pyramid ╱4, 1╲, with a mast which starts at a height of $1\frac{1}{2}$ above the point $2\frac{1}{2}$. Its Left boundary is the left side of this pyramid together with the mast. Now we compute $L_t(G) = R_t(H) - t$, $R_t(G) = L_t(K) + t$ (until G_t becomes a number) by pushing the Right boundary of H still further right, and the Left boundary of K still further left. Applied to the Right boundary of H this yields a line starting at 5 and travelling vertically upwards until $t = 1$, then diagonally right and up thereafter. From the Left boundary of K we get a line vertical till $t = 1\frac{1}{2}$, then diagonally up and left.

These lines meet at a height $t = 1\frac{3}{4}$ directly above the value $4\frac{1}{4}$, and so they define the Left and Right boundaries of G below this point, these boundaries above this point being vertical. So the diagram for G is a lop-sided "house" with a mast.

When we consider the implications of this procedure for the general short game G, we obtain:

THEOREM 61. *For any short game G, the thermograph is a region whose Left boundary is a line proceeding either vertically or diagonally up and right in stretches, the Right boundary being in stretches vertical or diagonal up and left. Beyond some point, both boundaries coincide in a single vertical line (the mast). The coordinates of all corners in the diagram are dyadic rationals.*

Proof. This requires only the observation that on subtracting t from a line which is vertical or diagonal up-and-left we obtain one correspondingly diagonal up-and-right or vertical, and that two such lines aiming towards each other must meet, at a point whose coordinates can be found with a single division by 2.

The proof of the theorem assures us at last that the definition of G_t has the properties presupposed in it, and incidentally makes Theorem 60 an honest theorem.

Now we ask about the corresponding sections $L(G_t)$ and $R(G_t)$. On which side are they of the numbers near to them?

THEOREM 62. (See Fig. 16). *The sections $L(G_t)$ and $R(G_t)$ are "just inside"* *the boundary of the diagram on vertical stretches, "just outside" on diagonal stretches. At points of the mast above its foot, $L(G_t)$ is to the right of $R(G_t)$ in the diagram; that is to say, $L(G_t) < R(G_t)$. At corners of the diagram the sections behave in the same way as at immediately smaller values of t. (So their behaviour is "continuous downwards".)*

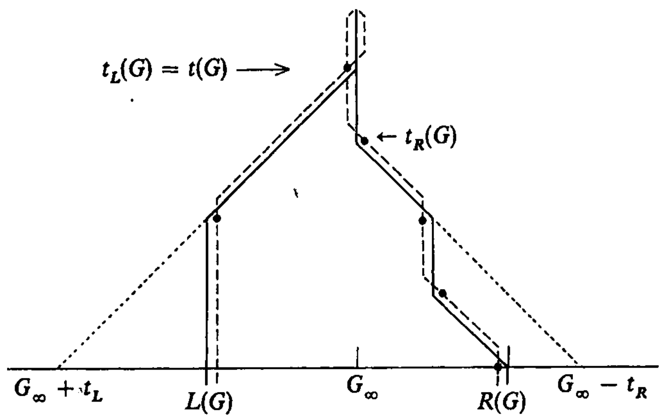


FIG. 16. The left and right sections of G_t are indicated by the dashed lines. Note how they cross the firm lines at corners, and cross each other at the foot of the mast. This behaviour is typical.

Proof. These properties are preserved in the passage from the diagrams for G^L and G^R to that for G.

Now Theorem 62 makes it natural to prolong the boundaries just a little way downward below the line $t = 0$. These prolongations are to be vertical when the corresponding section at $t = 0$ is just inside the thermograph diagram, and diagonally "outwards" when it is just outside. When we do this (as we shall), we read off the nature of the sections for $t = 0$ from the diagram as well. The rules for computing these prolongations are the obvious extensions of the rules for the rest of the diagram, and we shall say no more about them. The reader who examines Figs 15 and 16 closely will see that these prolongations were already present.

THEOREM 63.

$$G \geqslant x \text{ implies } G_t \geqslant x$$

$$(x + G)_t = x + G_t$$

$$(x - G)_t = x - G_t$$

for all short games G and dyadic rationals x.

Proof. Obvious from the properties and construction of thermographs.

THEOREM 64. $(G + H)_t = G_t + H_t$ *for short G, H.*

Proof. If G, H, or $G + H$ is equal to a number x, this follows from Theorem 63. Otherwise, we can use the inductive definitions of G_t, H_t, $(G + H)_t$ to give a 1-line proof:

$$G_t + H_t = \{G^L{}_t - t + H_t, G_t + H^L_t - t \mid G^R{}_t + t + H_t, G_t + H^R{}_t + t\}$$

$$= \{(G + H)^L_t \mid (G + H)^R_t\} = (G + H)_t.$$

THEOREM 65. *If* $G \geqslant H$, *then* $G_t \geqslant H_t$. *In particular, from* $G = H$, *we can deduce* $G_t = H_t$.

Proof. We have $G \geqslant H$ iff $G - H \geqslant 0$, so this theorem follows from the previous one.

Note. The contrary possibility that the value of G_t might depend on the *form* of G makes Theorems 63 and 64 slightly more subtle than they appeared at first sight. But all is now well.

Definition. We write G_∞ for the ultimate value of G_t, and t_L for the value of t beyond which $L(G_t) = L(G_\infty)$, t_R for the value beyond which $R(G_t) = R(G_\infty)$. The numbers t_L and t_R are called respectively the *Left and Right temperatures* of G, and their maximum is just *the* temperature $t(G)$ of G. See Fig. 16.

THEOREM 66. G_∞ *is none other than the mean value* $m(G)$ *of G. (From now on, we use the new notation* G_∞.) *We have the inequalities*

$$L(G_t) \leqslant L(G) \leqslant L(G_t) + t$$

$$R(G_t) - t \leqslant R(G) \leqslant R(G_t)$$

$$t(G + H) \leqslant \max(t(G), t(H))$$

(and similar inequalities with $t(G)$ *replaced by* $t_L(G)$, $t_R(G)$*), and also the equalities*

$$t_L(G) = t_R(-G), t(G) = t(-G),$$

and the "cooling equality"

$$(G_t)_u = G_{t+u}.$$

Proof. The first statement follows from Theorem 64 and the facts that $L(G_t) \leqslant L(G)$, $R(G) \leqslant R(G_t)$, which, like the remaining inequalities of the next two lines follow from the assertions about the slopes of the Left and Right boundaries. The third inequality is proved as follows: since for $t > t(G)$, $t(H)$ we have $G_t = G_\infty$, $H_t = H_\infty$, for such t we have

$$(G + H)_t = G_t + H_t,$$

a number. So such t are also greater than $t(G + H)$. The inequalities about $-G$ are obvious. So we are left with the cooling equality, which has a 1-line inductive proof.

This theorem implies in particular that we obtain the thermograph for G_t by *submerging* that for G to the depth t (see Fig. 17). In other words, the way we cool a game is by pouring cold water on it!

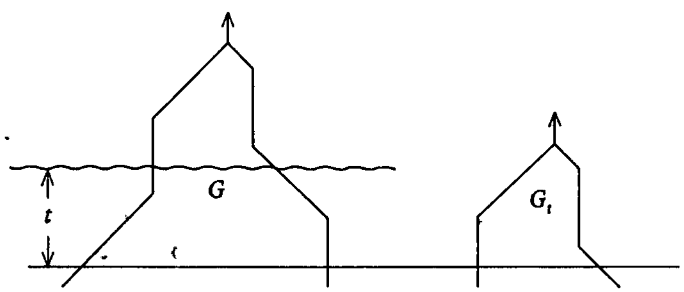


FIG. 17. How to cool a game by pouring water on it.

"Thermography" has been much extended and generalised by Elwyn Berlekamp and his co-workers, who have applied it to "Go" and other traditional games in the following works:

E. R. Berlekamp, Blockbusting and Domineering, *J. Combin. Theory Ser. A*, **49**(1988) 67–116.

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CHAPTER 10

Simplifying Games

You boil it in sawdust: you salt it in glue:
You condense it with locusts and tape:
Still keeping one principal object in view—
To preserve its symmetrical shape.

Lewis Carroll, "The Hunting of the Snark"

One quite valuable way to simplify games is to simplify our notation for them! (This is more important than it might seem, because even with the best will in the world, the names of games can get inordinately long.) So we first present some useful abbreviations.

We omit the curly brackets round games whenever this is possible without too much confusion—so for instance we shall write $A, B \mid C$ for the game $\{A, B \mid C\}$. Next, we need some way of distinguishing between $\{\{A \mid B\} \mid C\}$ and $\{A \mid \{B \mid C\}\}$, and so we introduce $\|$ as a 'stronger' separator than $|$, when these games become $A \mid B \,\|\, C$ and $A \,\|\, B \mid C$ respectively. ($A \,\|\, B \mid C$ may be pronounced "A slashes B slash C".) Thus the game we used as an example for temperature theory would now be called $7 \mid 5 \,\|\, 4 \mid 1$. Sometimes it is handy to introduce triple slashes $|||$, but usually we can get along quite happily with judicious use of brackets to supplement the above conventions.

The initial positions of many games are of the form

$$\{A, B, C, \ldots \mid -A, -B, -C, \ldots\}$$

being symmetrical as regards Left and Right. So we introduce the abbreviation $\pm(A, B, C, \ldots)$ for this game. In particular, the notation $\pm G$ will mean $\{G \mid -G\}$. Note that this will prevent us in future from using $\pm$ to denote an ambiguous sign, so that the phrase "$+x$ or $-x$" will appear more commonly than usual from now on. Finally, there are many positions of the form

$$\{A, B, C, \ldots \mid A, B, C, \ldots\}$$

in which the moves for Left and Right are *identical*, rather than *symmetrical*.

We shall use

$$\{A, B, C, \ldots\}$$

as an abbreviation for this game.

Some other notational conventions for particular games will be introduced in our Chapter 15. A fairly complete dictionary is given at the end of the book.

However, the real simplifications we have in mind concern the form of G rather than its name. The main problem is to see how we can simplify the *form* of a given game without affecting its *value*. We first discuss some modifications which might change the value, but in a predictable way.

THEOREM 67. *The value of G is unaltered or increased when we*
(*i*) *increase any G^L or G^R,*
(*ii*) *remove some G^R or add a new G^L,*
(*iii*) *replace the G^R by the K^R, for any game $K \geqslant G$.*

Proof. Let H be the game obtained by so modifying G. Then in the game $H - G$ it is easy to check that Right has no good first move.

Informally, it is even more obvious that these modifications are in Left's favour, for giving him new moves or prohibiting certain moves for Right will not harm Left. These principles are used repeatedly in analysing individual games, often in very much more general forms.

DOMINATED AND REVERSIBLE OPTIONS

Suppose two different Left options of G are comparable with each other, say $G^{L_1} \leqslant G^{L_0}$. Then we say G^{L_1} is *dominated* by G^{L_0}, since Left will plainly regard the latter as the better move. Similarly, if $G^{R_1} \geqslant G^{R_0}$ (note the reversed inequality) we call G^{R_1} dominated by G^{R_0}.

Now suppose instead that the Left option G^{L_0} has itself a Right option $G^{L_0R_0}$, say, for which we have the inequality $G^{L_0R_0} \leqslant G$. Then we say that the move from G to G^{L_0} is a *reversible* move, being reversible through $G^{L_0R_0}$. Similarly a Right option G^{R_1} of G is reversible (through $G^{R_1L_1}$) if and only if it has some Left option $G^{R_1L_1} \geqslant G$. It turns out that whenever one player (Left, say) makes a reversible move, his opponent might as well reverse it (for he improves on the original position by so doing). So instead of moving from G to G^{L_0}, Left might as well move straight from G to some $G^{L_0R_0L}$. A formal version of this result is part of the next theorem.

THEOREM 68. *The following changes do not affect the value of G.*
(*i*) *inserting as a new Left option any $A \lhd\!| \ G$, or as a new Right option any $B \ |\!\rhd G$.*

(*ii*) *Deleting any dominated option*
(*iii*) *If G^{L_0} is reversible through $G^{L_0R_0}$, replacing G^{L_0} as a Left option of G by all the Left options $G^{L_0R_0L}$ of $G^{L_0R_0}$.*
(*iv*) *If G^{R_1} is reversible through $G^{R_1L_1}$, similarly replacing G^{R_1} by all the $G^{R_1L_1R}$.*

Proof. Because of the importance of this theorem, we give a more detailed proof. Suppose first that $A \lhd\!| \ G$, and let $H = \{G^L, A \mid G^R\}$ be the modified game in (i). Then in $H - G$ the moves from H to G^L, G^R have as counters those from $-G$ to $-G^L$, $-G^R$, and conversely, and the move from H to A yields the position $A - G \lhd\!| \ 0$ by assumption. So there is no good move in $H - G$, whence $H = G$.

Part (ii) now follows, for if G^{L_1} is dominated by G^{L_0}, and H denotes G with G^{L_1} deleted, we have $G^{L_1} \leqslant G^{L_0} \lhd\!| \ H$, and so the insertion of G^{L_1} will not affect the value of H. Recall the fact that for any game G and any G^L, G^R, we have $G^L \lhd\!| \ G \lhd\!| \ G^R$, for from the difference $G - G^L$ or $G^R - G$, Left can plainly move to 0. (This theorem is part of Theorem 0 of part 0!)

Part (iii) is the most important and least obvious part. Let us write $G = \{G^{L_0}, G^{L'} \mid G^R\}$, $H = \{G^{L_0R_0L}, G^{L'} \mid G^R\}$, where $G^{L'}$ denotes the typical Left option other than G^{L_0} of G. Now consider the difference

$$H - G = \{G^{L_0R_0L}, G^{L'} \mid G^R\} + \{-G^R \mid -G^{L'}, -G^{L_0}\}.$$

The moves from H to $G^{L'}$ or G^R and from $-G$ to $-G^{L'}$, $-G^R$ counter each other, so we need only consider those from H to $G^{L_0R_0L}$ and from $-G$ to $-G^{L_0}$. The first of these is shown to be bad by

$$G^{L_0R_0L} \lhd\!| \ G^{L_0R_0} \leqslant G,$$

and the second is countered by the move from $-G^{L_0}$ to $-G^{L_0R_0}$, after which Right is to move in the position $H - G^{L_0R_0}$. His moves from $-G^{L_0R_0}$ to $-G^{L_0R_0L}$ have counters in H, so he must move from H to G^R. But this is a bad move, since $G^R - G^{L_0R_0} \geqslant G^R - G \ |\!\rhd 0$.

Part (iv) follows by symmetry.

THE SIMPLEST FORM OF A SHORT GAME

Now let G be a short game. We aim to find the simplest form of G. By induction, we can suppose that each game G^L, G^R has already been put into simplest form, if we like. In any case, we proceed as follows—eliminate from G any option which is dominated by some other option, and then replace any reversible option G^{L_0} or G^{R_1} by the corresponding smaller positions $G^{L_0R_0L}$ or $G^{R_1L_1R}$, respectively. Repeat, if necessary, until no option of G is dominated or reversible.

THEOREM 69. *Suppose that G and H (not necessarily short) have neither dominated nor reversible options. Then G and H are equal if and only if each Left or Right option of either is equal to a corresponding option (Left or Right respectively) of the other.*

Proof. Suppose $G = H$, and consider playing $G - H$. The move for Right to $G^R - H$ must have a reply for Left, say to either $G^{RL} - H$ or $G^R - H^R$. The former case is impossible, for it implies $G^{RL} \geqslant H = G$, so that G^R was reversible in G. So we have proved that for each G^R there is some H^R with $G^R \geqslant H^R$. Since similarly each $H^R \geqslant$ some G^R, and neither game has dominated options, we must in fact have each $G^R =$ some H^R and conversely. Similar statements hold for the Left options.

This theorem assures us that each short game has a unique *simplest form.* We shall now discuss some examples.

Examples. The position $\{\uparrow \mid \uparrow\}$. We know already that

$$\{\uparrow \mid \uparrow\} \geqslant \{0 \mid \uparrow\} = \uparrow + \uparrow + *$$

obviously greater than $* = \uparrow^R$. So $\uparrow$ is reversible through $*$ as a Left option, and can therefore be replaced by $*^L = 0$. So we have $\{\uparrow \mid \uparrow\} = \{0 \mid \uparrow\}$. Since there is no 0^R, 0 cannot be reversible in this (indeed, 0 can never be reversible in any game), and since $\{0 \mid \uparrow\}$ is positive (Left can win, Right can't), $\uparrow$ is not reversible as a Right option. So $\{\uparrow \mid \uparrow\} = \{0 \mid \uparrow\}$ in simplest form.
(Recall that $\uparrow$ is the game $\{0 \mid *\}$, where $* = \{0 \mid 0\}$.)

The game $x \mid y$. Let x and y be numbers, and consider the game $x \mid y$. Then if $x < y$, this is the simplest number between x and y, so we shall consider the contrary case $x \geqslant y$. Then plainly the game $x \mid y$ has no dominated options. Moreover, its thermograph is the pyramid $\diagup x, y \diagdown$, and so $x|y$ determines the numbers x and y. It must therefore be in simplest form, for if the option y (say) were reversible, we should have $x \mid y = x \mid y^{LR}$ or $\{x \mid \}$, which games have different thermographs.

Now we assert that for any number z, we have $(x \mid y) + z = (x + z \mid y + z.)$ This is because in the difference

$$(x \mid y) + z + (-y - z) \mid (-x - z)$$

the moves not in z have exact counters, while the move for Right (say) from z to z^R is countered by Left's move to $-y - z$, since by the thermograph, we have $x \mid y > y + (z - z^R)$.
This kind of translation invariance allows us to normalise $x \mid y$ to the form $u \pm v$, where $u = \frac{1}{2}(x + y)$, $v = \frac{1}{2}(x - y)$. Of course it holds only for $x \geqslant y$, and shows that in this region, $x \mid y$ exhibits a strikingly continuous behaviour for all real numbers x and y.

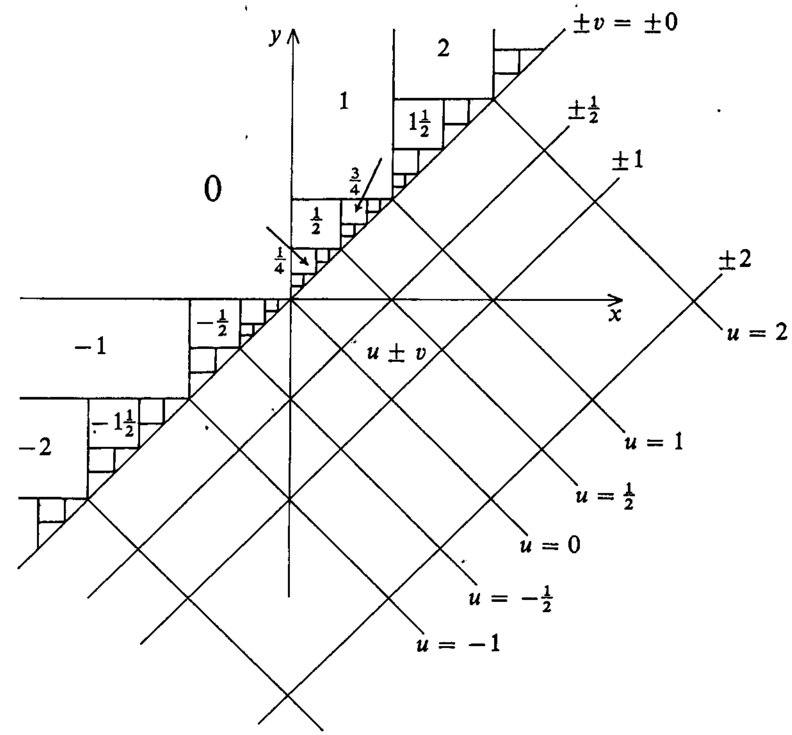


FIG. 18. The game $x \mid y$. Note: Points on boundaries here behave similarly to the points just South-East (↘) of them.

The game $x \pm a \pm b \pm \ldots \pm k$. Let $x, a, b, \ldots, k$ be numbers. Since $\pm t$ is its own negative, and is zero if $t < 0$, we can suppose that $a > b > c > \ldots > k \geqslant 0$. Then the thermograph of $\pm a \pm b$ is sketched in Fig. 19. This shows that $-a + b < \pm a \pm b < +a - b$. But this shows us that in the game

$$\pm a \pm b = \{a \pm b, \pm a + b \mid -a \pm b, \pm a - b\}$$

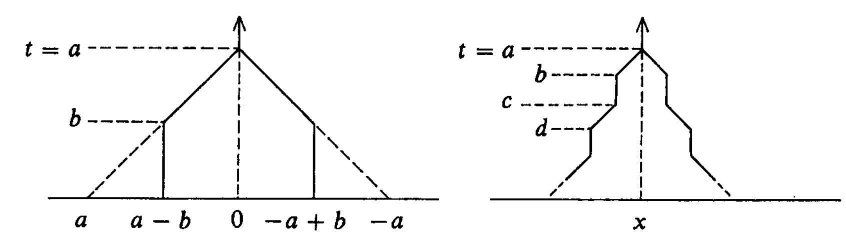


FIG. 19. Thermographs of $\pm a \pm b$ and $x \pm a \pm b \pm \ldots \pm k$.

the options $\pm a + b$ and $\pm a - b$ are dominated, for the difference between the two options on either side is just that between $\pm a \pm b$ and $a - b$. So in fact $\pm a \pm b = \{a \pm b \mid -a \pm b\}$, for since we know the simplest form $\{a + b \mid a - b\}$ of $a \pm b$, we can see that this option is not reversible. In a similar way, we find that the simplest form of $x \pm a \pm b \pm \ldots \pm k$ is

$$\{x + a \pm b \pm \ldots \pm k \mid x - a \pm b \pm \ldots \pm k\}$$

and that its thermograph is as shown.

Of course this uses the ordering $a > b > \ldots > k$, and is entirely concordant with experience and expectations. For since the game $\pm a$ represents an advantage of a move to the first player to move in it, when playing a sum of such games, the first player will take that with the largest a, then his opponent will take the next largest, and so on. In particular, the Left value will be $x + a - b + c - \ldots$, and the Right value $x - a + b - c + \ldots$.

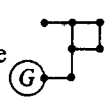
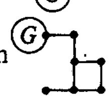
In practice it is often simpler not to normalise games $x \mid y$ to the form $u \pm v$, but the rules still apply—in a sum of such games one should always move in that with the largest diameter $x - y$. (The diameter as here defined is twice the temperature of this game.)

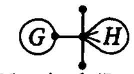
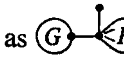
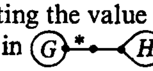
DOMINO POSITIONS AND PROPOSITIONS

We return to the game with dominoes discussed in Chapter 7. To avoid pages full of little squares, we represent positions by graphs in which nodes represent squares, and edges join nodes representing adjacent squares. (Compare our conventions for COL and SNORT.) In this form, Left's move is to remove two nodes joined by a vertical edge, while Right removes a pair of nodes joined by a horizontal edge.

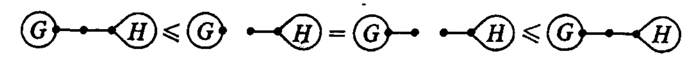
Note that the game could be played on any graph in which two kinds of edges are by definition called *horizontal* and *vertical*, but the addition of new such graphs does not seem to make the game any more interesting. Similar comments are often applicable to other games we shall discuss.

We attach at the end of the chapter a dictionary for dominoes like those for COL and SNORT. To show how the dictionary was prepared, we discuss in detail some of the results, and some particular positions. Most of the results referring to general positions are due to Norton.

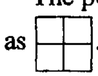
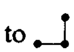
0. A graph like  (for instance) has the same value as the corresponding graph . (For the possible moves are in one-to-one correspondence.)

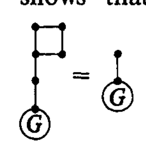
1. A position like  has the same value as . (For the two moves for Vertical (Left) through the central node are equivalent.)
2. If we delete a horizontal edge or introduce a new vertical edge, the value is unaltered or increased. (For these cannot harm Left or help Right.)
3. . (For the linking harms only Right.)
4. If the starred edge in  can be deleted without affecting the value of this position, then the same holds of the starred edge in .

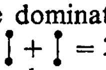
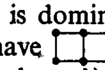
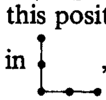
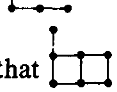
   (This edge is called *explosive*.) (From the inequalities

   .)

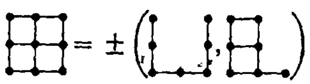
Now we discuss some particular positions.

The position  we have already discussed in Chapter 7, where it appeared as . Since the players have essentially unique moves, its value is plainly $\{1 \mid -1\} = \pm 1$. Now the position  has the same value, for the additional move for Left is to  (value $*$) which is dominated by the move to 1. This shows that the new edge is explosive, and so we have for instance

 ± 1.

In general, let us note that if Left has at most $n + 1$ moves, even supposing the collaboration of Right, and he actually has a first move leading to a position of value n, then this move dominates all others. In the position , for instance, Left's move to  $= 2$ is dominant, and Right has essentially only one move, to , so that we have  $= 2 \mid -\frac{1}{2} = \frac{3}{4} \pm 1\frac{1}{4}$. (The form $2 \mid -\frac{1}{2}$ is better in practice, $\frac{3}{4} \pm 1\frac{1}{4}$ 'in theory'.) So this position has mean value $\frac{3}{4}$. The two moves for Left (say) are equivalent in , so this position has value $\{-1 \mid 1\} = 0$. This result enables us to say that  has also the value $2 \mid -\frac{1}{2}$, and so the new edge in it is explosive.

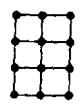
We are now in a position to evaluate the 3×3 square . (We should obviously describe sizes in terms of nodes here, since these correspond to squares in the original game.) In the position , the two moves for Left are to (0) and ($\frac{1}{2}$), while that for Right is to + (2). So $= \frac{1}{2} \mid 2 = 1$. Now we have the equation



$$= \pm\left(\ldots\right) = \{1, \tfrac{1}{2} \mid -2 \,\|\, -1, 2 \mid -\tfrac{1}{2}\} = G \text{ say.}$$

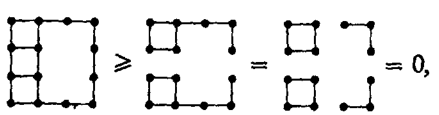
Now it is trivial that $-2 \leqslant G$ (add 2 to G and see how easy it is to win), so the Left option $\frac{1}{2} \mid -2$ is reversible through -2, and so can be replaced by the Left options (there aren't any) of -2. In this way, we see that G simplifies to ± 1, its simplest form.

It is not hard to show that the 3×4 rectangle has value $1\frac{1}{2}$. For this is plainly a lower bound (break the rectangle across the dotted line), and a quick strategic discussion shows that Left cannot win the difference



$$- 1\tfrac{1}{2}.$$

Larger rectangles are something of a problem. But if we only want to work out who wins, we can employ the following type of argument. From the 4×4 square, Left can move to



and so the 4×4 square is a win for the first player. Similar arguments can be found for the 4×6 rectangle, using the value of the position , which does not take too long to compute.

The 5×5 square can be shown to be a second player win (and so have value 0) by the following special strategy. This gives Left 6 moves, *or* keeps Right down to 5 moves and gives Left 5 moves.

Supposing Right's first move is in the top left 3×3 square, make the moves a, b of the first drawing if we can, followed by any c in the top three rows, and the moves d, e, f. If not, make the move b of the later drawings and occupy the centre if possible, followed by any move d *other* than e and f of the second drawing, then such of those moves e and f which are still legal. If this is impossible, the move c of the third drawing might be available, and lead back to a similar strategy to our first attempt.

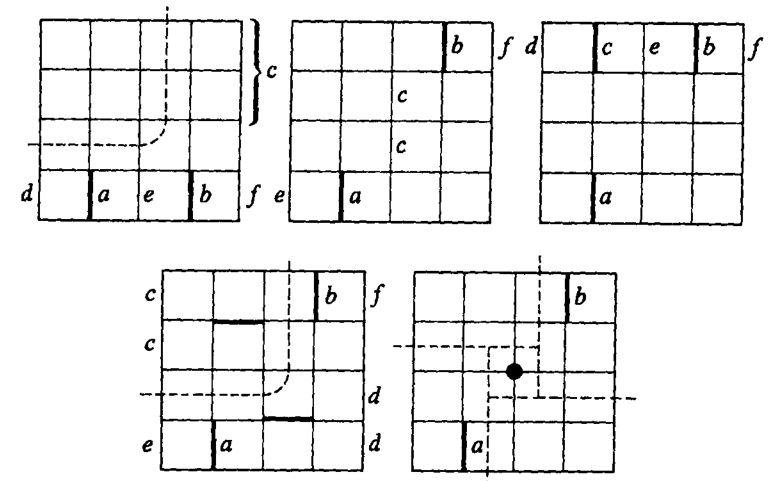


FIG. 20. The equation $5 \times 5 = 0$.

If not, the position is *either* as in the fourth drawing, with Right's third move in the top left 3×3 square, say, when we can make the moves of that drawing, *or* Right has taken the centre. In the last case, we do not really break any horizontal lines when we partition as indicated into 4 regions of values $1, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$, and so we are at least half a move ahead.

The zig-zag patterns $\cdot$, , , , ,... give rise to an interesting sequence of values. Letting ZZ_n be the value of the n-square zig-zag pattern, we find $ZZ_1 = 0$, $ZZ_2 = 1$, $ZZ_3 = *$, $ZZ_4 = 1 \mid 0$, $ZZ_5 = \pm 1$, $ZZ_6 = 2 \mid *$, $ZZ_7 = \pm 1*$, $ZZ_8 = 2 \mid 1 \,\|\, 0$, $ZZ_9 = \pm(2 \mid 0, 2 \mid *)$, $ZZ_{10} = 1*$, and $ZZ_{11} = ZZ_9*$. The later values get more complicated, but we can fairly easily calculate them *almost exactly*.

In fact we find

$$ZZ_{8n+1 \text{ or } 8n+3} = 0\text{-ish}$$
$$ZZ_{8n+1 \text{ or } 8n-3} = \pm 1\text{-ish}$$
$$ZZ_{8n+2} = 1\text{-ish}$$
$$ZZ_{8n-2} = 2 \mid 0\text{-ish}$$
$$ZZ_{4n} = (n \mid n-1 \,\|\, n-2 \,|||\, n-3 \ldots 1 \,|||||||||\, 0)\text{-ish}$$

where the suffix "-ish" means "infinitesimally shifted". In other words, we write G-ish for $G + \varepsilon$, when ε is infinitesimal. In these particular cases, of course, the various infinitesimal shifts ε are small games.

The game ZZ_{4n} has mean value $1-(1/2^n)$, is strictly less than 1, and strictly greater than any negative number, but not greater than 0. These results follow from the thermographs:

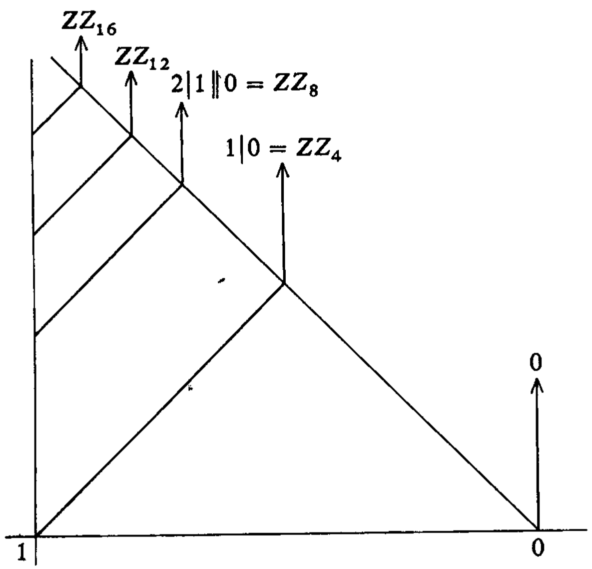
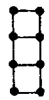
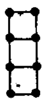
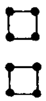
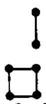
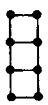


FIG. 21.

The rectangle  has a very interesting value. Note first that we have  $\geqslant$  $= 0$, so the value is zero or positive. The moves for Left are to  and , which equal  (0) and  $(2|0)$ by some of our theorems. The moves for Right are to  $(0|-2)$ and  $(\frac{1}{2}|-2)$. So we have the equality

 $= \{0, 2|0\|0|-2, \frac{1}{2}|-2\} = G$, say.

The option $\frac{1}{2}|-2$ is plainly dominated by $0|-2$, and since $G \geqslant 0$, the Left option $2|0$ is reversible to 0, and can be replaced by the (non-existent) 0^L. So we have $G = 0\|0|-2$, which, since we see it is strictly positive, is in simplest form.

For reasons that will only appear later, this game is called $+_2$ (pronounced "tiny-two"). For any positive number x, we have a similar game "tiny-x"

$$\text{"tiny-}x\text{"} = 0\|0|-x = +_x.$$

For each positive x, $+_x$ is a positive infinitesimal, and indeed a *small* game in the sense of Chapter 9, since it is strictly smaller than all positive numbers. But other calculations show that for strictly positive x these games are smaller than all positive all small games, such as $\uparrow$. As a matter of notation, we abbreviate sums involving such games in a natural way—thus $5+_2$ means $5 + +_2$, and $5-_2$ means $5 - +_2$.

It is possible to define powers $\uparrow^x$ of $\uparrow$ for positive $x \geqslant 1$ so that whenever $x > y$, then $\uparrow^x$ is infinitesimal compared to $\uparrow^y$, and all these powers are all small. We have thus a rough-and-ready scale of infinitesimals:

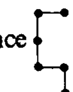
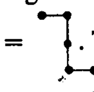
Firstly, infinitesimal numbers, like $1/\omega$, $1/\varepsilon_0$, etc.

Next, the all small games, such as $\uparrow$, $\uparrow^2$, etc.

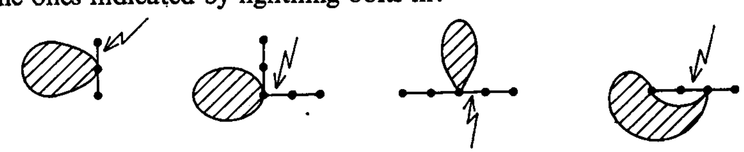
Finally, the games like $+_1$, $+_2$, etc.

We say *finally* because indeed the games $+_\alpha$ really tend to zero as α tends to **On**, any strictly positive game being bigger than some $+_\alpha$. (Any *short* positive game is greater than some $+_n$.) But we should also add that, *zerothly*, there are some infinitesimal games that are strictly greater than all infinitesimal numbers! These remarks are very much amplified in Chapter 16.

A DOMINO DICTIONARY

We now tabulate values for all domino positions with at most 6 nodes (Fig. 22). The game of dominoes has a behaviour in some ways intermediate between the two games COL and SNORT of this chapter, with typical values not so restricted as those of COL nor so chaotic as those of SNORT. Many of these are derivable from each other by simple rules. Often it obviously does not affect play if we make a configuration bend in the opposite direction to the given one—for instance  = . There are also a number of rules telling us that on certain occasions edges may be deleted without affecting values, as described earlier in this chapter. Here is a brief catalogue of explosive edges:

The ones indicated by lightning bolts in:



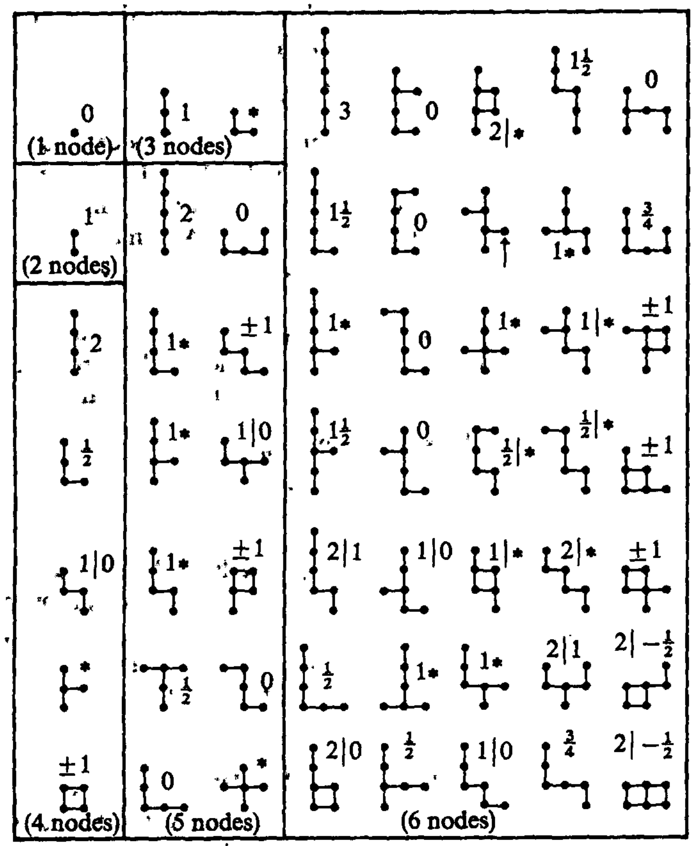
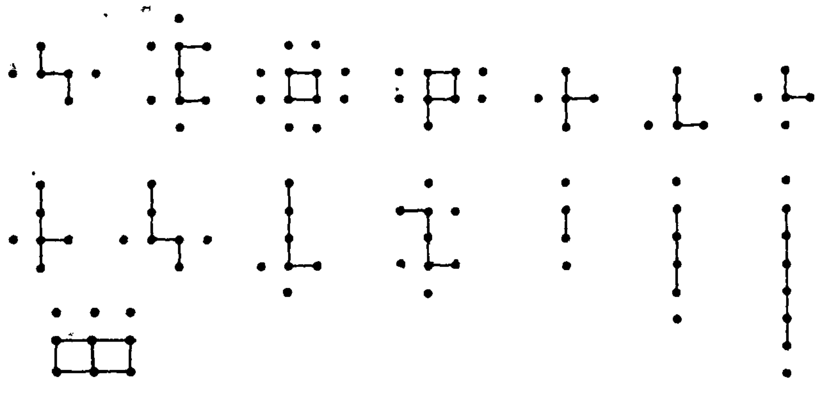
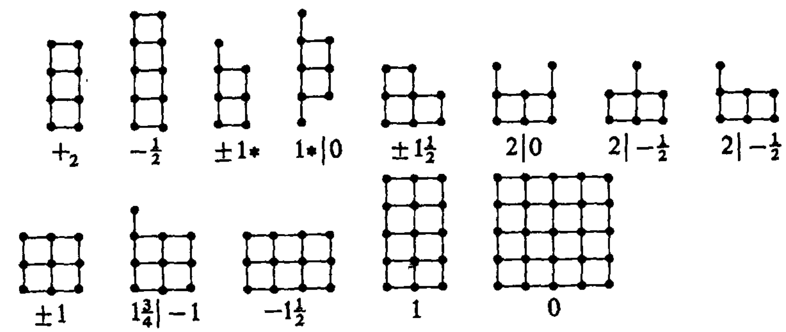


FIG. 22.

and edges joining any one of the configurations below to any one of the indicated surrounding nodes:



Some fairly large domino positions we have analysed are:



We have chosen these as being of shapes fairly likely to arise in actual play.

Göran Andersson's Domino game is called "Domineering" in *Winning Ways*, where larger dictionaries can be found. The necessary evaluations have been greatly eased by David Wolfe's "combinatorial games toolkit" for partizan game theory which runs on Linux computers. You can obtain it from

http://www.gustavus.edu/~wolfe/papers-games/

or by sending e-mail to wolfe@gustavus.edu.