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A NOTE ON PARTITIONS AND TRIANGLES WITH INTEGER SIDES

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In a recent paper [2], Jordan, Walch, and Wisner characterize the number $T(n)$ of incongruent triangles with integer sides that have perimeter n. They determine $T(n)$ by first noting that $T(4)=0$, $T(6)=T(8)=1$, $T(10)=2$, $T(12)=3$, $T(14)=4$, and then proving two theorems equivalent to the assertions: (1) $T(2n+12)=T(2n)+n+3$; (2) $T(2n)=T(2n-3)$. In this note we remark that $T(n)$ may be simply handled by relating it to $p_3(n)$ and $p_2(n)$, the number of partitions of n into 3 and 2 parts, respectively. In the following $[x]$ denotes the greatest integer in x and $\{x\}$ is the nearest integer to x.

THEOREM. $T(n)=p_3(n)-\sum_{1<j<\left[\frac{1}{2}n\right]} p_2(j).$

Proof. Each partition of n into three parts yields a unique triangle of the desired type and conversely, except when the sum of the smallest two parts does not exceed the largest part. This happens for each partition of j into two parts c and d with $1<j<\frac{1}{2}n$, for then $c+d+(n-j)$ is the related partition of n and $c+d<n-j$. Hence $T(n)=p_3(n)-\sum_{1<j<[\frac{1}{2}n]}p_2(j)$.

COROLLARY. $T(n)=\left\{\frac{n^2}{12}\right\}-\left[\frac{n}{4}\right]\left[\frac{n+2}{4}\right].$

Proof. Since $p_2(n)=[\frac{1}{2}n]$ (see [1, p. 81, Ex. 1]), it is a simple problem in mathematical induction to prove that

$$\sum_{1<j<\frac{1}{2}n} p_2(j)=\left[\frac{n}{4}\right]\left[\frac{n+2}{4}\right].$$

The formula $p_3(n)=p(\text{“}\{1,2,3\}\text{”},n-3)=\{n^2/12\}$ for $n>0$ is given in Example 2 of [1, p. 81].

We note that this corollary gives us an operational formula through which we may easily compute $T(n)$; furthermore, all the assertions for $T(n)$ described in the first paragraph are easily deduced from it.

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