The average number of i-dimensional implicants

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February 17, 2011

Conjecture 0.1. \( \text{ImplicantsAve}(n, a, i) = \left( \frac{a}{2^i} \right)^{2^n-i} \left( \frac{n}{i} \right) \left( \frac{2^n}{2^i} \right) \)

Proof. For each implicant of dimension \( i > 0 \), it came from 2 implicants of dimension \( i-1 \) such that these two implicants can be expressed as two sets of identical elements except one and the one with different signs. For example:

\[
v_1 = \{x_1, \ldots, x_k, \ldots, x_{n-i+1}\} \\
v_2 = \{y_1, \ldots, y_k, \ldots, y_{n-i+1}\}
\]

where \( x_i = y_i, i = 1, \ldots, n - i + 1, i \neq k \) and \( x_k = -y_k \).

Therefore, an implicant of dimension \( i \) came from \( 2^i \) implicants of dimension 0 such that these \( 2^i \) implicants (expressed as sets) contain \( n-i \) identical elements and \( i \) elements with different combinations of signs. Let \( S \) be a random set of 0-dimensional implicants with \( n \) variables and \( |S|=a \), we get that

\[
\text{ImplicantsAve}(n, a, i) = \sum_{s_1, \ldots, s_{2^i} \in S} \text{Prob}(s_1, \ldots, s_{2^i} \text{can be combined to one implicant})
\]

Since \( s_1, \ldots, s_{2^i} \) share \( n-i \) identical elements and \( i \) elements with different combination of signs, the probability in the above formula is \( \frac{2^{n-i} \binom{n}{i}}{\binom{2^n}{2^i}} \) where \( \binom{2^n}{2^i} \) is the total number of ways to choose \( 2^i \) different sets, \( \binom{n}{i} \) is the number of
ways to choose the $i$ elements with different combinations of signs and $2^{n-i}$ is the number of the combinations of $n-i$ identical elements. Finally, we get that

$$ImplicantsAve(n, a, i) = \left( \frac{a}{2^i} \right) \frac{2^{n-i} \binom{n}{i}}{\binom{2^n}{2^i}}$$