1 Introduction

• Linguistic aspects of Programming.
• Programming best practices
• Program writing comparative analysis
• Experimental Programming and it’s Algorithmic aspects.
• Emphasis on Symbolic Mathematics, (Algorithms, Numerics) and available Platforms

2 Design of Experiments

The goal of this section is to establish the theoretical framework we will use in the context of our experimental setup. I will introduce the framework as well as the particular conjectures and hypotheses in number theory that our experimental setup will aim to understand and address.

2.1 An Identity Involving the Zeta function

We note that

\[ \zeta(n) = \left( \sum_{k \in \mathbb{N}} \frac{1}{k^n} \right) = \sum_{0 \leq t \leq p - 1} \left( \sum_{k \equiv t \mod(p)} \frac{1}{k^n} \right) \]  

\[ \Rightarrow \left( 1 - \frac{1}{p^n} \right) \zeta(n) = \sum_{k \not\equiv 0 \mod(p)} \frac{1}{k^n} \]  

by multiplying both sides of the equation by \((\zeta(n))^{m} \cdot \frac{p^n}{n!}\) we get:

\[ \frac{\zeta^{m+1}(n)}{n!} \left( x^n - \frac{x^n}{p^n} \right) = \sum_{1 \leq k \leq p - 1} \frac{\zeta^{m}(x)}{n!} \left( \frac{x}{k} \right)^n \]
and by summing over the values of \( n \) we have:

\[
\sum_{1 \leq n \leq \infty} \frac{\zeta^{m+1}(n)}{n!} \left( x^n - \frac{x^n}{p^n} \right) = \sum_{1 \leq n \leq \infty} \left( \sum_{1 \leq k \leq p-1} \frac{\zeta^m(n)}{k^n} \right) \left( \frac{x^n}{x^k} \right) \]

(4)

\[
\Rightarrow \sum_{1 \leq n \leq \infty} \frac{\zeta^{m+1}(n)}{n!} \left( x^n - \frac{x^n}{p^n} \right) = \sum_{1 \leq n \leq \infty} \left( \sum_{1 \leq k \leq p-1} \frac{\zeta^m(n)}{k^n} \right) \frac{x^n}{n!} \]

(5)

When \( m = 1 \) we have

\[
\sum_{1 \leq n \leq \infty} \frac{\zeta^2(n)}{n!} \left( x^n - \frac{x^n}{p^n} \right) = \sum_{1 \leq n \leq \infty} \left( \sum_{1 \leq k \leq p-1} \frac{\zeta(n)}{k^n} \right) \frac{x^n}{n!} \]

(6)

by interchanging the order of the sum we obtain

\[
\sum_{1 \leq n \leq \infty} \frac{\zeta^2(n)}{n!} \left( x^n - \frac{x^n}{p^n} \right) = \sum_{1 \leq k \leq p-1} \left( \sum_{1 \leq n \leq \infty} \frac{\zeta(n)}{n!} \left( \frac{x^n}{x^k} \right) \right) \]

(7)

Furthermore we observing that

\[
\left( \sum_{0 < k < p} \frac{\zeta(n)}{n!} \left( \frac{x^n}{x^k} \right) \right) = \sum_{0 < k < p} e^{x/n \cdot k} \]

(8)

It follows that

\[
\sum_{1 \leq n \leq \infty} \frac{\zeta^2(n)}{n!} \left( x^n - \frac{x^n}{p^n} \right) = \sum_{0 < k < p} e^{x/n \cdot k} \]

(9)

from which we deduce that

\[
\lim_{p \to \infty} \left\{ \sum_{0 < k < p} e^{x/n \cdot k} \right\} = \sum_{1 \leq n \leq \infty} \frac{\zeta^2(n)}{n!} x^n \]

(10)

the next case corresponds to \( m = 2 \), we have
\[
\sum_{1 \leq n \leq \infty} \frac{\zeta^3(n)}{n!} \left( x^n - \frac{x^n}{p^n} \right) = \sum_{1 \leq k \leq p-1} \left( \sum_{1 \leq n \leq \infty} \frac{\zeta^2(n)}{n!} \left( \frac{x}{k} \right)^n \right)
\]

(11)

from the our result we obtain

\[
\sum_{1 \leq n \leq \infty} \frac{\zeta^3(n)}{n!} \left( x^n - \frac{x^n}{p^n} \right) = \sum_{1 \leq k \leq p-1} \left( \sum_{0 < s < \infty} \sum_{0 \leq t \leq \infty} e^{s/k \cdot t} \right)
\]

(12)

similarly we obtain that

\[
\lim_{p \to \infty} \left\{ \sum_{0 < k < p} \sum_{0 \leq s < \infty} \sum_{0 \leq t \leq \infty} e^{s/k \cdot t} \right\} = \sum_{1 \leq n \leq \infty} \frac{\zeta^3(n)}{n!} x^n
\]

(13)

From which we deduce that

\[
\sum_{1 \leq n \leq \infty} \frac{\zeta^m(n)}{n!} x^n = \sum_{0 < i_1, \ldots, i_m < \infty} \exp \left\{ \frac{x}{i_1 \times \cdots \times i_m} \right\}
\]

(14)

2.2 Harmonic Power Sum from the Euler \( \gamma \) constant

Starting from Euler's famous constant \( \gamma \) defined by

\[
\gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n} - \ln(n) \right)
\]

(15)

\[
\gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \ln \left( e^{n-1} \right) - \ln(n) \right)
\]

(16)

\[
\gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \ln \left( n^{-1} \cdot e^{n-1} \right) \right)
\]

(17)

\[
\gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \left[ \ln \left( \frac{d}{dt} e^{t-1} \right) \right]_{t=0} \right)
\]

(18)

The constant \( \gamma \) leads us naturally to the harmonic power sums by considering the function \( g_n(x) \) defined by
\[ g_n(x) = \left( \sum_{1 \leq k \leq n-1} e^{\frac{x}{k}} \right) + x \cdot \ln \left( n^{-1} \cdot e^{n-1} \right) - n \quad (19) \]

\[ \Rightarrow g_n(x) = \left( \sum_{1 \leq k \leq n-1} e^{\frac{x}{k}} \right) + \ln \left( n^{-x} \cdot e^{x} \right) - n \quad (20) \]

\[ \Rightarrow g_n(x) = \ln \left( \frac{e^{(-n+x)}}{n^x} \right) + \sum_{1 \leq k \leq n-1} e^{\frac{x}{k}} \quad (21) \]

So as to yield the following Taylor series

\[ g(x) = \lim_{n \to \infty} g_n(x) = 1 + \gamma x + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k!} x^k \quad (22) \]

By abuse of notation the series

\[ \sum_{k \in N} \frac{\zeta(k)}{k!} x^k \quad (23) \]

will refer to the functions \( g(x) \) even though we know that \( \zeta(0) \) and \( \zeta(1) \) are not defined.

### 2.3 Special relation between two kinds of series

Let us consider two functions \( f(z) \) and \( g(z) \) of respective power series expansion

\[ f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k \]

\[ g(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k \]

our aim in this section is to express the function \( g(z) \) in terms of \( f(z) \) for this we must recall that

\[ \frac{1}{\Gamma(x)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it} \left( -it \right)^{-x} dt \]

so we have:

\[ \sum_{k=1}^{\infty} \frac{a_k}{\Gamma(k)k} z^k = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \sum_{k=1}^{\infty} \frac{a_k}{k} \left( \frac{z}{-it} \right)^k \right] e^{it} dt \]

So we deduce that:
For $0 < z < \infty$ the preceding corresponds to a Fourier transform expressed by:

$$g(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \sum_{k=1}^{\infty} \frac{a_k}{k} \left( \frac{z}{-it} \right)^k \right] e^{it} dt$$

$$\Rightarrow g(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f\left( \frac{z}{it} \right) e^{it} dt$$

$$\Rightarrow g(z) = \frac{1}{2\pi} \int_{\frac{s}{z} = -\infty}^{\frac{s}{z} = \infty} f\left( \frac{1}{1-is} \right) e^{isz} ds$$

An immediate consequence is that:

$$\left( e^{-ix} - 1 \right) H(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -\log \left\{ \left( 1 + \frac{1}{it} \right)^x \right\} e^{ixt} dt$$

which is verifiable by the known Fourier identity:

$$\left( e^{-ix} - 1 \right) H(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -\log(1 + \frac{1}{it}) e^{ixt} dt$$

For the discussion to be complete let us discuss the transform that should carry us from $g(x)$ to $f(x)$ as defined earlier.

$$f(x) = \sum_{k=1}^{\infty} \frac{a_k}{k} x^k = \sum_{k=1}^{\infty} (k-1)! \frac{a_k}{k!} x^k$$

but we recall that $\Gamma(k) = (k-1)! = \int_0^\infty t^{k-1} e^{-t} dt$ so we have:

$$f(x) = \sum_{k=1}^{\infty} \frac{a_k}{k} x^k = \sum_{k=1}^{\infty} \int_0^\infty t^{k-1} e^{-t} dt \frac{a_k}{k!} x^k$$

$$\Rightarrow f(x) = \int_0^\infty \left[ \sum_{k=1}^{\infty} \frac{a_k}{k!} (xt)^k \right] \frac{1}{t} e^{-t} dt$$

from which we deduce that

$$f(x) = \int_0^\infty g(xt) \frac{1}{t} e^{-t} dt$$

furthermore
\[ f(x) = \int_{t=0}^{t=\infty} g(xt) \frac{x}{t} e^{-\frac{x}{t}} d\left(\frac{tx}{x}\right) \]

\[ \Rightarrow f(x) = \int_{t=\frac{z}{x}}^{t=\frac{\infty}{x}} g(\frac{tx}{x}) \frac{x}{t} e^{-\frac{x}{t}} d\left(\frac{tx}{x}\right) \]

for \(0 < x < \infty\) we have

\[ f(x) = \int_{q=0}^{q=\infty} \frac{1}{q} g(x^2 q) e^{-qx} dq \]

### 2.4 Harmonic power sums and series

we define the harmonic power sum \(G_n(z)\) to be the function

\[ G_n(z) = \sum_{k=1}^{n} z^k \]

the harmonic power sums are more difficult to handle than geometric sums. Following from the preceding section we have:

\[ G_n(z) = \frac{1}{2\pi} \int_{t=-\frac{(n-1)}{\pi}}^{t=\frac{(n-1)}{\pi}} \frac{1}{(n-1)} \prod_{k=1}^{n} (e^{it\frac{1}{(n-1)}} - z^\frac{1}{(n-1)}) e^{-it} dt \]

or alternatively we may write:

\[ \Rightarrow G_n(z) = \frac{1}{2\pi} \int_{t=-\frac{(n-1)}{\pi}}^{t=\frac{(n-1)}{\pi}} \left( \frac{z}{(n-1)} \prod_{k=1}^{n} (e^{it\frac{1}{(n-1)}} - z^\frac{1}{(n-1)}) \right) e^{-it} dt \]

Furthermore we have

\[ G_n(e^x) = \frac{1}{2\pi} \int_{x=\frac{(n-1)}{\pi}}^{x=\frac{(n-1)}{\pi}} \left( \frac{x}{(n-1)} \prod_{k=1}^{n} (e^{it\frac{1}{(n-1)}} - e^{i\frac{1}{(n-1)}}) \right) e^{-it} d\left(\frac{q}{x}\right) \]

by posing \(t = \frac{q}{x}\)

\[ \Rightarrow G_n(e^x) = \frac{1}{2\pi} \int_{t=-\frac{(n-1)}{\pi}}^{t=\frac{(n-1)}{\pi}} \left[ \frac{x}{(n-1)} \prod_{k=1}^{n} (e^{it\frac{1}{(n-1)}} - e^{it\frac{1}{(n-1)}}) \right] e^{-itx} dt \]

incidentally

\[ \Rightarrow G_n(e^{ix}) = \frac{1}{2\pi} \int_{t=-\frac{(n-1)}{\pi}}^{t=\frac{(n-1)}{\pi}} \left[ \frac{x}{(n-1)} \prod_{k=1}^{n} (e^{it\frac{1}{(n-1)}} - e^{it\frac{1}{(n-1)}}) \right] e^{-itx} dt \]
By Taylor expansion we get the following:

\[
\lim_{n \to \infty} \left\{ G_{n-1} \left( e^x \right) + \ln \left( n^{-x} \cdot e^{\frac{x}{n}} \right) - n \right\} = \lim_{n \to \infty} \left\{ \left( \sum_{k=1}^{n-1} e^{\frac{x}{k}} \right) - \log \left( n^x \right) - n \right\} = \sum_{k \in \mathbb{N}} \frac{\zeta(k)}{k!} x^k
\]

where

\[
\sum_{k \in \mathbb{N}} \frac{\zeta(k)}{k!} x^k \equiv \left( 1 + \gamma x + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} x^k \right).
\]

We recall from complex analysis that:

\[
\log \left( e \cdot \Gamma \left( 1 - x \right) \right) = \sum_{k \in \mathbb{N}} \frac{\zeta(k)}{k} x^k, \quad |x| < 1
\]

We therefore deduce that

\[
\left( \lim_{n \to \infty} \left\{ G_{n-1} \left( e^x \right) + \ln \left( n^{-x} \cdot e^{\frac{x}{n}} \right) - n \right\} \right) H(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \log \left( e \cdot \Gamma \left( 1 + \frac{1}{it} \right) \right) x e^{-i\pi t} dt
\]

One of the important aims of this section is to provide a reasonable approximation of the series renormalized or regularized harmonic power series expressed by

\[
\lim_{n \to \infty} \left\{ \left( \sum_{k=1}^{n-1} e^{\frac{x}{k}} \right) - \log \left( n^x \right) - n \right\} \equiv \sum_{k \in \mathbb{N}} \frac{\zeta(k)}{k!} x^k
\]

clearly for very small values of \( x \) that is

\[
|x| \ll 1
\]

we have

\[
\lim_{n \to \infty} \left\{ \left( \sum_{k=1}^{n-1} e^{\frac{x}{k}} \right) - \log \left( n^x \right) - n \right\} \approx \log \left( e \cdot \Gamma \left( 1 - x \right) \right)
\]

unfortunately this approximation fails us for values of \( x \) that are closer to 1 lets us briefly discuss by how much this approximation fails us for this we compute

\[
\log \left( e \cdot \Gamma \left( 1 - x \right) \right) - \lim_{n \to \infty} \left\{ \left( \sum_{k=1}^{n-1} e^{\frac{x}{k}} \right) - \log \left( n^x \right) - n \right\} = \left( \sum_{k \in \mathbb{N}} \frac{\zeta(k)}{k!} x^k \right) - \left( \sum_{k \in \mathbb{N}} \frac{\zeta(k)}{k!} x^k \right)
\]

\[
= \left( \sum_{k \in \mathbb{N}} \frac{\zeta(k)}{k!} x^k \right) \left( \frac{1}{k} - \frac{1}{k!} \right) x^k
\]
\[
\frac{1}{m} - \frac{1}{n} = \sum_{k \in \mathbb{N}} \zeta(k) \left( \frac{(k-1)!}{k!} - 1 \right) x^k
\]  

(30)

From which we deduce that

\[
\lim_{n \to \infty} \left\{ \left( \sum_{k=1}^{n-1} e^x \right) - \log (n^x) - n \right\} = \log (e \cdot \Gamma (1 - x)) + \left( \sum_{k \in \mathbb{N}} \zeta(k) \left( \frac{(k-1)! - 1}{k!} \right) x^k \right)
\]

(31)

a straightforward lower and upper bound estimates is given by

\[
\log (e \cdot \Gamma (1 - x)) + \left( \sum_{3 \leq k \leq \infty} \zeta(k) \left( \frac{(k-1)! - 1}{k!} \right) x^k \right) \geq \left\{ \left( \sum_{k=1}^{n-1} e^x \right) - \log (n^x) - n \right\} \geq \left\{ \left( \sum_{k=1}^{n-1} e^x \right) - \log (n^x) - n \right\} \geq \log (e \cdot \Gamma (1 - x)) + \left( \sum_{3 \leq k \leq \infty} \zeta(k) \left( \frac{(k-1)! - 1}{k!} \right) x^k \right)
\]

(32)

\[\Rightarrow 2 \log (e \cdot \Gamma (1 - x)) - \left( 1 + \gamma x + \frac{\pi x^2}{6} \right) - e^x + \left( 1 + x + \frac{x^2}{2} \right) \geq \left\{ \left( \sum_{k=1}^{n-1} e^x \right) - \log (n^x) - n \right\} \geq 2 \log (e \cdot \Gamma (1 - x)) - \left( 1 + \gamma x + \frac{\pi x^2}{6} \right) - e^{-x} \left( 1 - x + \frac{x^2}{2} \right)
\]

(33)

for

\[f_1(x) = 2 \log (e \cdot \Gamma (1 - x)) - \left( 1 + \gamma x + \frac{\pi x^2}{6} \right) - e^x + \left( 1 + x + \frac{x^2}{2} \right)
\]

(34)

\[f_2(x) = 2 \log (e \cdot \Gamma (1 - x)) - \left( 1 + \gamma x + \frac{\pi x^2}{6} \right) + e^{-x} - \left( 1 - x + \frac{x^2}{2} \right)
\]

(35)

\[\left\{ \left( \sum_{k=1}^{n-1} e^x \right) - \log (n^x) - n \right\} \approx \frac{f_1(x) + f_2(x)}{2}
\]

(36)

that is to say that

\[\left\{ \left( \sum_{k=1}^{n-1} e^x \right) - \log (n^x) - n \right\} \approx 1 + (1 - \gamma) x - \left( \frac{\pi^2}{6} \right) x^2 - \sinh(x) + 2 \log (\Gamma (1 - x))
\]

(37)

appears to be a reasonable approximation of the regularized series

\[\left\{ \left( \sum_{k=1}^{n-1} e^x \right) - \log (n^x) - n \right\} \equiv \sum_{k \in \mathbb{N}} \zeta(k) \frac{x^k}{k!}
\]

(38)
2.4.1 Some Notes

Before we return to the harmonic power sum we discuss a useful application of Fourier analysis. In all generality we have

\[ P(x) = \sum_{1 \leq k \leq n} a_{f(k)} x^{f(k)} = \frac{1}{2\pi} \int_{s=-\pi}^{s=\pi} \left( \prod_{1 \leq k \leq n} \left( e^{is} - a_{f(k)} x^{f(k)} \right) \right) e^{-is(n-1)} \, ds \]  

(39)

\[ = -\frac{1}{2\pi} \int_{s=-\pi}^{s=\pi} \left( \prod_{1 \leq k \leq n} \left( 1 - a_{f(k)} x^{f(k)} \cdot e^{-is} \right) \right) e^{is} \, ds \]  

(40)

We may also derive the following identity assuming that \( \{\alpha_k\} \subset \mathbb{N}^* \)

\[ \left( -\frac{1}{2\pi} \int_{t=-\infty}^{t=\infty} \prod_{1 \leq k \leq n} \left( 1 + e^{x+is} \frac{1}{1-\alpha_k} \right) \alpha_k^m \cdot e^{is} \, ds \right) = \sum_{1 \leq k \leq n} \alpha_k^m \cdot e^x \alpha_k^{m-1} \]  

(41)

which follows from the binomial identity

\[ \left( -\frac{1}{2\pi} \int_{t=-\infty}^{t=\infty} \prod_{1 \leq k \leq n} \left( 1 + e^{x+is} \frac{1}{1-\alpha_k} \right) \alpha_k^m \cdot e^{is} \, ds \right) = \sum_{1 \leq k \leq n} \alpha_k^m \cdot e^x \alpha_k^m \]  

(42)

\[ \Rightarrow -\frac{1}{2\pi} \int_{t=-\infty}^{t=\infty} \int_{s=0}^{s=2\pi} \prod_{1 \leq k \leq n} \left( 1 + e^{t+is} \frac{1}{1-\alpha_k} \right) \alpha_k^m \cdot e^t \cdot ds \cdot dt = \sum_{1 \leq k \leq n} e^x \alpha_k^m \]  

(43)

We set \( z = t + is \Rightarrow dz = dt + i \cdot ds \)

\[ \Rightarrow -\frac{1}{2\pi} \int_{t=-\infty}^{t=\infty} \int_{s=0}^{s=2\pi} \prod_{1 \leq k \leq n} \left( 1 + e^{t+is\alpha_k^{1-\alpha_k^m}} \right) \alpha_k^m \cdot e^t \cdot ds \cdot dt = \sum_{1 \leq k \leq n} e^x \alpha_k^m \]  

(44)

We recall from Green’s theorem that

\[ \int_C f(z) \, dz = \int \int_R \left( \frac{\partial (i \cdot f(z))}{\partial t} - \frac{\partial (f(z))}{\partial s} \right) \, ds \cdot dt \]  

(45)

so for \( f(z) \) solution to the partial differential equation

\[ \frac{\partial (i \cdot f(z))}{\partial t} - \frac{\partial (f(z))}{\partial s} = e^z \left( \prod_{1 \leq k \leq n} \left( 1 + \sqrt{e^{(z+\alpha_k^m)}(1-\alpha_k^{1-\alpha_k^m})} \right) \alpha_k^m \right) \]  

(46)
We may stretch the green theorem to write the following equality.

\[ \int_{\gamma} f(z) \, dz = \sum_{1 \leq k \leq n} e^{x \cdot \alpha_k^m} \]  

(47)

in particular for \( \alpha_k = k, 1 \leq k \leq n \), the expression above completely determines sums of the form

\[ \sum_{1 \leq k \leq n} x^k \]  

(48)

I shall later think about this expression in relation with the jacobitheta function.

The preceding can be thought of as some trick allowing us to view finite sum of terms as fourier transform we discuss here another illustration of the same principle.

\[ \sum_{1 \leq k \leq n} x^k = \frac{1}{2\pi} \int_{t=-\pi}^{t=\pi} \left( \prod_{1 \leq k \leq n} (e^{i t} - x^k) \right) e^{-i t (n-1)} \, dt = \left( \frac{x^{n+1} - x}{x - 1} \right) \]  

\[ \Rightarrow \sum_{1 \leq k \leq n} x^k = \frac{1}{2\pi} \int_{t=-\pi}^{t=\pi} \left( \prod_{1 \leq k \leq n} (1 - x^k \cdot e^{-i t}) \right) e^{i t} \, dt = \left( \frac{x^{n+1} - x}{x - 1} \right) \]  

As a direct consequence we have:

\[ \sum_{1 \leq k \leq n} e^{i k x} = 2^{n-1} \left( \frac{i^n}{\pi} \right) \int_{t=-\pi}^{t=\pi} \left( \prod_{1 \leq k \leq n} \sqrt{x} (k - t) \frac{x}{2} \right) e^{i t} \, dt = \sqrt{x} (n+1) \frac{x}{2} \left( \sin \left( \frac{\pi}{2} \right) \right) \]  

Furthermore

\[ \sum_{1 \leq k \leq n} x^{s \cdot k} = \frac{1}{2\pi} \int_{t=-\pi}^{t=\pi} \left( \prod_{1 \leq k \leq n} (1 - x^{s \cdot k} \cdot e^{-i t}) \right) e^{i t} \, dt = \left( \frac{x^{s \cdot (n+1)} - x^s}{x^s - 1} \right) \]  

It follows that another experimental approach for investigating the harmonic power progression consists in looking at the transform

\[ \sum_{1 \leq k \leq n} x^{\frac{1}{s} \cdot k} = \frac{1}{2\pi} \int_{s=-\pi}^{s=\pi} \left( \prod_{1 \leq k \leq n} (e^{i s} - x^{\frac{1}{s}}) \right) e^{-i (n-1) s} \, ds \]  

(49)

\[ \sum_{1 \leq k \leq n} x^{\frac{1}{s} \cdot k} = \frac{1}{2\pi} \int_{s=-\pi}^{s=\pi} \left( 1 - x^{\frac{1}{s}} \cdot e^{-i s} \right) e^{-i s} \, ds \]  

(50)

our attempt to characterize the preceding integral, we start with the expression.
\[
\prod_{k=1}^{n} (e^{is}k - x) = \prod_{1 \leq k \leq n} \left( \prod_{1 \leq \ell \leq k} (e^{is} - x \, e^{i \frac{2\pi \ell}{k}}) \right)
\]

We note that

\[
(e^{is} - x) \prod_{k=2}^{n} (e^{is}k - x) = \prod_{1 \leq k \leq n} \left( \prod_{1 \leq \ell \leq k} (e^{is} - x \, e^{i \frac{2\pi \ell}{k}}) \right)\]

\[
\Rightarrow \prod_{1 \leq k \leq n} (e^{is} - x \frac{1}{k} \, e^{i \frac{2\pi}{k}}) = \frac{\prod_{t=1}^{n} (e^{i \cdot t \cdot s} - x) \prod_{2 \leq k \leq n} \left( \prod_{2 \leq t \leq k} (e^{is} - x \, e^{i \frac{2\pi \ell}{k}}) \right)}{2 \leq k \leq n \, 2 \leq t \leq k} (51)
\]

we introduced the expression above because it relates the harmonic power progression to the partition function.

\[
\sum_{1 \leq k \leq n} x^{\frac{1}{k}} \cdot e^{i \frac{2\pi}{k}} = -\frac{1}{2\pi} \int_{s=-\pi}^{s=\pi} \prod_{1 \leq k \leq n} (e^{is} - x \frac{1}{k} \, e^{i \frac{2\pi}{k}}) \, e^{-i(n-1)s} \, ds \quad (52)
\]

\[
\sum_{1 \leq k \leq n} x^{\frac{1}{k}} \cdot e^{i \frac{2\pi}{k}} = -\frac{1}{2\pi} \int_{s=-\pi}^{s=\pi} \prod_{2 \leq k \leq n} \left( \prod_{2 \leq t \leq k} (e^{is} - x \, e^{i \frac{2\pi \ell}{k}}) \right) e^{-is(n-1)} \, ds \quad (53)
\]

Equivalently we may write

\[
\Rightarrow \sum_{1 \leq k \leq n} x^{\frac{1}{k}} \cdot e^{i \frac{2\pi}{k}} = -\frac{1}{2\pi} \int_{s=-\pi}^{s=\pi} \left( \prod_{2 \leq k \leq n} \left( 1 - x \left( e^{-is} \right)^{1} \right) \prod_{2 \leq t \leq k} \left( 1 - x \, e^{i \frac{2\pi \ell}{k} - s} \right) \right) e^{is} \, ds \quad (54)
\]

The expression above relates quite naturally to Euler’s pentagonal identity as follows

\[
\prod_{n=1}^{\infty} (1 - y^n) = \sum_{k \in \mathbb{Z}} (-1)^k \, y^{\frac{3k-1}{2}} \quad (55)
\]

by performing the change of variable \( t = e^{-is} \) in the expression of Euler’s pentagonal identity

\[
\prod_{n=1}^{\infty} (1 - e^{-isn}) = \sum_{k \in \mathbb{Z}} (-1)^k \, e^{-ik \frac{3k-1}{2}} \quad (56)
\]
alternatively we write

$$\Rightarrow \prod_{n=1}^{\infty} (1 - e^{-i\pi n}) = \sum_{k \in \mathbb{Z}} e^{-ik\left(\frac{3k \cdot s - s}{2} + k \cdot \pi\right)}$$  \hspace{1cm} (57)$$

$$\Rightarrow \prod_{n=1}^{\infty} (1 - e^{-i\pi n}) = 2 \sum_{k \in \mathbb{N}} \cos\left(k \frac{3k \cdot s - s}{2} + k \cdot \pi\right)$$  \hspace{1cm} (58)$$

One would want to consider the following identity resulting from the choice \( x = e^0 \)

$$\Rightarrow \sum_{1 \leq k \leq \infty} e^{i\pi} = -\frac{1}{2\pi} \int_{\pi}^{3\pi} \left( \prod_{2 \leq k \leq n} \left(1 - e^{-i\pi} \right) \frac{\prod_{2 \leq k \leq n} \left(1 - e^{i(2\pi t - s)} \right)}{2 \leq t \leq k} \right) e^{is} ds$$  \hspace{1cm} (59)$$

and as \( n \to \infty \) we have

$$\sum_{1 \leq k \leq \infty} e^{i\pi} = -\frac{1}{\pi} \int_{\pi}^{3\pi} \left( \prod_{2 \leq k \leq \infty} \left(1 - e^{-i\pi} \right) \frac{\prod_{2 \leq k \leq \infty} \left(1 - e^{i(2\pi t - s)} \right)}{2 \leq t \leq k} \right) e^{is} ds$$  \hspace{1cm} (60)$$

which yields

$$\sum_{1 \leq k \leq \infty} e^{i\pi} = -\frac{1}{\pi} \int_{\pi}^{3\pi} \left( \prod_{2 \leq k \leq \infty} \left(1 - e^{-i\pi} \right) \frac{\prod_{2 \leq k \leq \infty} \left(1 - e^{i(2\pi t - s)} \right)}{2 \leq t \leq k} \right) e^{-is} ds$$  \hspace{1cm} (61)$$

If one aims to investigate the following more general expression

$$\sum_{1 \leq k \leq n} e^{i\pi} = -\frac{1}{\pi} \int_{\pi}^{3\pi} \left( \prod_{2 \leq k \leq n} \left(1 - e^{-i\pi} \right) \frac{\prod_{2 \leq k \leq n} \left(1 - x \left(e^{-i\pi} \right)^k \right)}{2 \leq t \leq n} \frac{\prod_{2 \leq k \leq n} \left(1 - x \left(e^{i(2\pi t - s)} \right) \right)}{2 \leq t \leq k} \right) e^{-is} ds.$$  \hspace{1cm} (62)$$

The expression suggest the investigation of the partition function
\[
\sum_{m,n} P_{m,n} x^m y^n = \prod_k \left( 1 - x \cdot y^k \right)^{-1}
\]

(63)

\[
= \prod_k \left( \sum_l x^l \cdot y^{k \cdot l} \right)
\]

(64)

In the expression \( P_{m,n} \) counts the number of partition of the number \( n \) in \( m \) parts.

\[
\sum_{1 \leq k \leq n} x^k \cdot e^{i \frac{2\pi}{n} x} = -\frac{1}{2\pi} \int_{s=-\pi}^{s=\pi} \left( \sum_{u,v} P_{u,v} x^u \left( e^{-is} \right)^{-1} \right) \frac{\prod_{2 \leq k \leq n} \left( 1 - x \cdot e^{i\left(2\pi k - s\right)} \right)}{\prod_{2 \leq t \leq k} \left( 1 - x \cdot e^{i\left(2\pi t - s\right)} \right)} e^{-is} ds.
\]

(65)

We may compare trigonometric function resulting from the geometric progression and the harmonic power progression.

\[
\sum_{1 \leq k \leq n} e^{i \frac{2\pi}{n} x} = -\frac{1}{2\pi} \int_{s=-\pi}^{s=\pi} e^{is} \prod_{1 \leq k \leq n} \left( 1 - e^{i\left(\frac{2\pi k - s}{k}\right)} \right) ds
\]

(66)

\[
= -2^{n-1} \left( \frac{i^n}{\pi} \right) \int_{s=-\pi}^{s=\pi} e^{is} \prod_{1 \leq k \leq n} \sqrt{e^{i\left(\frac{2\pi k - s}{k}\right)}} \sin \left( \frac{2\pi x - k \cdot s}{2k} \right) ds
\]

(67)

and

\[
\sum_{1 \leq k \leq n} e^{i \cdot k \cdot x} = -2^{n-1} \left( \frac{i^n}{\pi} \right) \int_{s=-\pi}^{s=\pi} \prod_{1 \leq k \leq n} \sqrt{e^{i\left(k-s\right)x}} \cdot \sin \left( \left( k - s \right) \frac{x}{2} \right) e^{i\cdot s} ds
\]

(68)

I can compare the two expression by considering the points on the \((n+1)\) dimensional sphere of radius 1 having as coordinate \(y_{n+1}\) respectively equal to

\[
y_{n+1} = \prod_{1 \leq k \leq n} \sin \left( \frac{2\pi x - k \cdot s}{2k} \right)
\]

(69)

\[
y_l = \prod_{1 \leq k \leq l-1} \sin \left( \frac{2\pi x - k \cdot s}{2k} \right) \cdot \cos \left( \frac{2\pi x - l \cdot s}{2l} \right), \quad 1 < l < n + 1
\]

(70)
\[ y_1 = \cos \left( \pi \cdot x - \frac{s}{2} \right) \]  

(71)

for the harmonic power sum and the geometric progression we have

\[ y_{n+1} = \prod_{1 \leq k \leq n} \sin \left( (k - s) \cdot \frac{x}{2} \right) \]  

(72)

\[ y_l = \left( \prod_{1 \leq k \leq l-1} \sin \left( (k - s) \cdot \frac{x}{2} \right) \right) \cdot \cos \left( (l - s) \cdot \frac{x}{2} \right) , \quad 1 < l < n + 1 \]  

(73)

\[ \cos \left( (1 - s) \cdot \frac{x}{2} \right) \]  

(74)

We can associate to each one of these points a probability distribution and quantify the difference through their corresponding entropy.

Given the combinatorial interpretation of the product \( \prod_{n=1}^{m} (1 - x t^n) \).

Our way to characterize that product would be through the relationship between symmetric polynomials we consider

\[ P(z) = \prod_{n=1}^{m} (z - x \cdot t^n) = z^m + \sum_{0 \leq k \leq m-1} a_k(t, x) \cdot z^k \]  

(75)

so that

\[ P(1) = \prod_{n=1}^{m} (1 - x \cdot t^n) = 1 + \sum_{0 \leq k \leq m-1} a_k(t, x) \]  

(76)

The power of the roots is given by

\[ c_l(t, x) = \left( \sum_{0 \leq n \leq m} (xt^n)^2 \right) = \left( x^l \sum_{0 \leq n \leq m} (t^n)^2 \right) = x^l \left( \frac{1 - t^{(m+1)}}{1 - t^l} \right) \]  

(77)

\[
\begin{bmatrix}
a_{m-1}(t, x) \\
a_{m-2}(t, x) \\
\vdots \\
a_1(t, x) \\
a_0(t, x)
\end{bmatrix}
= -\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & c_1(t, x) & 2 & 0 & \cdots & 0 \\
0 & c_2(t, x) & c_1(t, x) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & c_{m-1}(t, x) & \cdots & c_2(t, x) & c_1(t, x) & m
\end{bmatrix}^{-1}
\begin{bmatrix}
c_1(t, x) \\
c_2(t, x) \\
c_3(t, x) \\
\vdots \\
c_m(t, x)
\end{bmatrix}
\]  

(78)

In particular we have

\[ a_{m-1} = x \left( \frac{1 - t^{(m+1)}}{1 - t} \right) \]  

(79)
\[ a_0(x,t) = \left(-x \cdot t^{\frac{m+1}{2}}\right)^m \]  (80)

The expression above provides us with an algorithm for determining the partitions of given size for a number. In order to characterize the general expression we consider

Furthermore we have

\[ Q(z) = \prod_{1 \leq n \leq \infty} (z - x \cdot t^n) = \sum_{0 \leq k \leq \infty} \alpha_k(t,x) \cdot z^k \]  (81)

so that

\[ Q(1) = \prod_{1 \leq n \leq \infty} (1 - x \cdot t^n) = \sum_{0 \leq k \leq \infty} \alpha_k(t,x) \]  (82)

The power of the roots is given by

\[ \kappa_l(t,x) = \left( \sum_{0 \leq n \leq \infty} (xt^n)^l \right) = \left( x^l \sum_{0 \leq n \leq \infty} (t^n)^l \right) = \frac{x^l}{1-t^l} \]  (83)

Let \( g_x(s) \) be

\[ g_x(s) = \sum_{1 \leq n \leq \infty} \left( \frac{x^n}{1-x^n} \right) \left( \frac{s^n}{n!} \right) \]  (85)

we note that

\[ \left[ \left( \frac{d}{ds} \right)^n g_x(s) \right]_{s=0} = \left( \sum_{1 \leq t \leq \infty} (x^n t^l) \right) = \left[ \left( \frac{d}{ds} \right)^n \sum_{1 \leq t \leq \infty} e^{x^t s} \right]_{s=0} \]  (86)

\[ \Rightarrow \sum_{1 \leq n \leq \infty} \left[ \left( \frac{d}{ds} \right)^n g_x(s) \right]_{s=0} \frac{s^n}{n!} = \sum_{1 \leq n \leq \infty} \left[ \left( \frac{d}{ds} \right)^n \sum_{1 \leq t \leq \infty} e^{x^t s} \right]_{s=0} \frac{s^n}{n!} \]  (87)

from which we deduce that
Our next goal is to try to determine the following sum

\[ f_x(s) = \sum_{1 \leq n \leq \infty} \left( \sum_{1 \leq t \leq \infty} (x^n)^t \right) \frac{s^n}{n} \]  

we recall that

\[ f_x(s) = \int_0^\infty g_x(s \cdot y) \left( \frac{e^{-y}}{y} \right) dy \]

by the discussion in section || furthermore by interchanging the order of summation we have

\[ -f_x(s) = - \sum_{1 \leq t \leq \infty} \left( \sum_{1 \leq w \leq \infty} \frac{(s \cdot x^t)^n}{n} \right) \]  

\[ = - \sum_{1 \leq t \leq \infty} \ln \left( 1 - s \cdot x^t \right) \]  

\[ -f_x(s) = \ln \left\{ \prod_{1 \leq t \leq \infty} \left( \frac{1}{1 - s \cdot x^t} \right) \right\} \]

So that one obtains the following identity

\[ \exp \{-f_x(s)\} = \exp \left\{ - \sum_{1 \leq t \leq \infty} \int_0^\infty e^{(s-1) y x^t} \frac{dy}{y} \right\} = \sum_{m, n} \mathcal{P}_{m, n} s^m x^n \]

2.5 Dual distributions with respect to differential operators

Let us now discuss particular class of distributions in relation to differential operators. In order to illustrate our point let us consider the following sums:

\[ f_p(x) = \sum_{1 \leq k \leq p} e^{k \cdot x} \]  

and

\[ g_p(x) = \sum_{1 \leq k \leq p} e^{x/k} \]

we have

\[ \left( \frac{d}{dx} \right)^m f_p(x) = \sum_{1 \leq k \leq p} k^m \cdot e^{x \cdot k} \]
for convenience let us define for a suitably chose distribution \( l(x) \) the following operator:

\[
\left( \frac{d}{dx} \right)^{-1} l(x) \equiv \int_{-\infty}^{x} l(t)dt
\]  

and incidentally

\[
\left( \frac{d}{dx} \right)^{-m} f_p(x) = \sum_{1 \leq k \leq p} k^{-m} e^{x-k}
\]

The very important fact about these two families of distribution resides in the following identity for any positive or negative integers :

\[
\left[ \left( \frac{d}{dx} \right)^{-m} f_p(x) \right]_{x=0} = \left[ \left( \frac{d}{dx} \right)^{m} g_p(x) \right]_{x=0}
\]

More generally for some suitably chosen constant \( c \) and for \( \{ \alpha_k \} \subset \mathbb{C}^* \) we have

\[
f_p(x) = \sum_{1 \leq k \leq p} a_k \cdot e^{\alpha_k (x-c)}
\]

and

\[
g_p(x) = \sum_{1 \leq k \leq p} a_k \cdot e^{\frac{x-c}{\alpha_k}}
\]

these two distributions are dual functions with respect to differential operators at the point \( c \)

\[
\left[ \left( \frac{d}{dx} \right)^{-m} f_p(x) \right]_{x=c} = \left[ \left( \frac{d}{dx} \right)^{m} g_p(x) \right]_{x=c}
\]

For formal series expressions of the form

\[
f(x) = \lim_{p \to \infty} f_p(x) = \sum_{1 \leq k \leq \infty} a_k \cdot e^{\alpha_k (x-c)},
\]

and

\[
g(x) = \lim_{p \to \infty} g_p(x) = \sum_{1 \leq k \leq \infty} a_k \cdot e^{\frac{x-c}{\alpha_k}}.
\]

In special cases of course the convergence of such series is completely determined by the coefficient \( a_k \), given the convergence of the series \( \sum_{k=1}^{\infty} |a_n| \), according to the Weierstrass test, both series corresponding to \( f(x) \) and \( g(x) \) converge uniformly for \( (x - c) < 0 \) and we have

\[
\left[ \left( \frac{d}{dx} \right)^{-m} f(x) \right]_{x=c} = \left[ \left( \frac{d}{dx} \right)^{m} g(x) \right]_{x=c}
\]
Let us now discuss some examples of dual functions with respect to differential operators. Consider the following integrals

\[ f(x) = \int_0^1 e^{xt} \, dt \quad (106) \]

\[ g(x) = \int_0^1 e^{\xi t} \, dt \quad (107) \]

To convince ourselves that these integrals are indeed dual let us use the Riemann integrals expression induced by these two integrals

\[ f_p(x) = \frac{1}{p} \sum_{1 \leq k \leq p} \exp \left\{ x \cdot \left( \frac{k}{p} \right) \right\} \quad (108) \]

\[ g_p(x) = \frac{1}{p} \sum_{1 \leq k \leq p} \exp \left\{ \frac{x}{\left( \frac{k}{p} \right)} \right\} \quad (109) \]

It is clear from their expressions that the functions \( f_p(x) \) and \( g_p(x) \) are dual functions at the point \( x = 0 \) incidentally the induced functions

\[ f(x) \equiv \lim_{p \to \infty} \left\{ f_p(x) \equiv \frac{1}{p} \sum_{1 \leq k \leq p} \exp \left\{ x \cdot \left( \frac{k}{p} \right) \right\} \right\} = \int_0^1 e^{xt} \, dt = \frac{(e^x - 1)}{x} \quad (110) \]

\[ g(x) \equiv \lim_{p \to \infty} \left\{ g_p(x) \equiv \frac{1}{p} \sum_{1 \leq k \leq p} \exp \left\{ \frac{x}{\left( \frac{k}{p} \right)} \right\} \right\} = \int_0^1 e^{\xi t} \, dt = (e^x + x \Gamma (0, -x)) \text{ if } \Re(x) \leq 0 \quad (111) \]

where \( \Gamma (a, x) \) denotes the incomplete gamma function. We therefore summarize our first example by the following equation

\[ \left[ \left( \frac{d}{dx} \right)^m \frac{(1 - e^x)}{x} \right]_{x=0} = - \left[ \left( \frac{d}{dx} \right)^m \left( e^x + x \Gamma (0, -x) \right) \right]_{x=0} \quad (112) \]

it may be pointed out that the example induces a family of dual functions given by

\[ f_y(x) \equiv \lim_{p \to \infty} \left\{ f_p(x) \equiv \frac{1}{p} \sum_{1 \leq k \leq p} \exp \left\{ x \cdot \left( \frac{k}{y} \right) \right\} \right\} = \int_0^1 e^{xt} \, dt = \frac{(e^{xy} - 1)}{x} \quad (113) \]
and

\[ g_p(x) \equiv \lim_{p \to \infty} \left\{ g_p(x) \equiv \frac{1}{p} \sum_{1 \leq k \leq p} \exp \left\{ \frac{x}{y \cdot \frac{k}{p}} \right\} \right\} = \]

\[ \int_0^y e^{\frac{t}{y}} dt = \left( ye^{x/y} + xT \left( 0, -\frac{x}{y} \right) \right) \text{ if } \Re(x) \leq 0 \text{ and } \Re(x) \geq 0 \quad (114) \]

The method above yields an efficient algorithm for a special kind of definite integrals. The last example we will discuss here is provided by one of Ramanujan beautiful identities namely

\[ \frac{d\pi(x)}{dx} = \left( \frac{1/y}{\ln (x)} \right) \sum_{1 \leq k \leq \infty} \frac{\mu(r)}{r} x^r \quad (115) \]

\[ = \left( \sum_{1 \leq k \leq \infty} \frac{\mu(r)}{r} x^r \right) \quad (116) \]

\[ \Rightarrow \frac{d}{dx} \frac{\pi(x)}{\ln (\ln (x))} = \sum_{1 \leq k \leq \infty} \frac{\mu(r)}{r} x^r \quad (117) \]

\[ \Rightarrow \frac{d\pi(x)}{d\ln (\ln (x))} = \frac{d\pi(e^t)}{dx} = \sum_{1 \leq k \leq \infty} \frac{\mu(r)}{r} x^r \quad (118) \]

Note that

\[ \Rightarrow \frac{d\pi(x)}{d\ln (\ln (x))} = \frac{d\pi(e^t)}{dt} \sum_{1 \leq k \leq \infty} \frac{\mu(r)}{r} e^{\frac{t}{r}} \quad (119) \]

where \( t = \ln (\ln (x)) \). \( \mu \) denotes the mobius function, we shall also investigate the mobius transform. \( \pi(x) \) is the function which gives the number of primes less than the number \( x \).

\[ g(e^x) = \sum_{1 \leq k \leq \infty} \frac{\mu(r)}{r} e^{\frac{x}{r}} \quad (120) \]

Let \( g(x) \) be a function such that

\[ f(x) = \sum_{1 \leq k \leq \infty} \frac{\mu(r)}{r} x^r \quad (121) \]

From which it follows that the functions

\[ g(e^x) = \sum_{1 \leq k \leq \infty} \left( \frac{\mu(r)}{r} \right) e^{\frac{x}{r}} \quad (122) \]
and

\[ f(e^x) = \sum_{1 \leq k \leq \infty} \left( \frac{\mu(r)}{r} \right) e^{\lambda r} \]

are dual to one another with respect to differential operators at the point \( x = 0 \)

\[
\left[ \frac{d}{dx} \right]^{-m} \left[ \sum_{1 \leq k \leq \infty} \left( \frac{\mu(r)}{r^m} \right) e^{\lambda r} \right]_{x=0} = \left[ \left( \frac{d}{dx} \right)^{m} \frac{d\pi(e^x)}{d\ln(\ln(x))} \right]_{x=0} \tag{124}
\]

\[
\left[ \sum_{1 \leq k \leq \infty} \left( \frac{\mu(r)}{r^{m+1}} \right) e^{\lambda r} \right]_{x=0} = \left[ \left( \frac{d}{dx} \right)^{m} \frac{d\pi(e^x)}{d\ln(x)} \right]_{x=0} \tag{125}
\]

From which it follows that

\[
\left[ \sum_{1 \leq k \leq \infty} \left( \frac{\mu(r)}{r^{m+1}} \right) e^{\lambda r} \right]_{x=0} = \left[ \left( e^{-t} \frac{d}{dt} \right)^{m} \frac{d\pi(e^t)}{dt} \right]_{t=0} \tag{126}
\]

### 2.6 The Partition function and Analytical Continuation

Let us consider the following algebraic equation

\[
X^n + \sum_{1 \leq k \leq n} (-1)^k S_k \cdot X^{n-k} = 0 \tag{127}
\]

By the fundamental theorem of algebra we know that the equations has \( n \) roots here denoted \( \{X_k\}_{1 \leq k \leq n} \) it follows

\[
X_1^n + \sum_{1 \leq k \leq n} (-1)^k S_k \cdot X_1^{n-k} = 0
\]

\[
X_2^n + \sum_{1 \leq k \leq n} (-1)^k S_k \cdot X_2^{n-k} = 0
\]

\[\vdots\]

\[
X_n^n + \sum_{1 \leq k \leq n} (-1)^k S_k \cdot X_n^{n-k} = 0 \tag{128}
\]

By summing over the equation

\[\sigma_n + \left( \sum_{1 \leq k \leq n-1} (-1)^k S_k \cdot \sigma_{n-k} \right) + (-1)^n S_{n} \cdot n = 0 \tag{129}\]

Let us choose the following convention \( S_0 = 1 \) and \( \sigma_{0n} = n \) so the equation above becomes

\[S_0 \cdot \sigma_n + \left( \sum_{1 \leq k \leq n-1} (-1)^k S_k \cdot \sigma_{n-k} \right) + (-1)^n S_{n} \cdot \sigma_{0n} = 0 \tag{130}\]
From which it follows that
\[
\sum_k \left( \sum_{i+j=k} (-1)^i S_i \cdot \sigma_j \right) X^k = 0
\]  
(131)
furthermore we note that
\[
\sum_k \left( \sum_{i+j=k} (-1)^i S_i \cdot \sigma_j \right) X^k
\]  
(132)
We consider the expression
\[
\left( \sum_i S_i \cdot (-t)^i \right) \left( \sum_j (X \cdot t)^j \right) = \sum_n a_n(X) \cdot t^n
\]  
(133)
where
\[
a_n(X) = S_0 \cdot X^n + \sum_{1 \leq k \leq n} (-1)^k S_k \cdot X^{n-k}
\]  
(135)
Which is the expression of the polynomial we started out with, for convinience we rewrite the equation as follows
\[
a_n(X) = \left( \sum_{i+j=n} (-1)^i S_i \cdot X^j \right) = 0, \quad 0 \leq i, j \leq n
\]  
(136)
From which we have
\[
\left( \sum_i S_i \cdot (-t)^i \right) \left( \sum_j (X \cdot t)^j \right) = \sum_i \left( S_i \cdot (-t)^i \right) \frac{1}{1 - X \cdot t} = \sum_n a_n(X) t^n = \langle a(X), t \rangle = 0
\]  
(137)
some observation
\[
\sum_{i+j=n} (-1)^i S_i \cdot (X_\mu)^j = 0 = \sum_{i+j=n} (-1)^i S_i \cdot \sigma_j
\]  
(138)
\[(X_\mu)^j \rightarrow \sigma_j
\]  
(139)
And the equation138 for different values of \(n\) yields the relation between the elementary symmetric polynomials and the Newton symmetric polynomials express by the following equations
\[( -1)^0 S_0 \cdot \sigma_n = - \left[ \sum_{1 \leq k \leq n-1} (-1)^k S_k \cdot \sigma_{n-k} \right] + (-1)^n S_n \cdot \sigma_0 \] 

(140)

\[( -1)^0 S_0 \cdot \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_{n-1} \\ \sigma_n \end{bmatrix} = - \begin{bmatrix} \sigma_0 & 0 & 0 & \cdots & 0 \\ \sigma_1 & \sigma_0 & 0 & \cdots & 0 \\ \sigma_2 & \sigma_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \sigma_{n-1} & \cdots & \sigma_2 & \sigma_1 & \sigma_0 \end{bmatrix}^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_{n-1} \\ \sigma_n \end{bmatrix} = \begin{bmatrix} (-1)^1 S_1 \\ (-1)^2 S_2 \\ \vdots \\ (-1)^{n-1} S_{n-1} \\ (-1)^n S_n \end{bmatrix} \] 

(141)

\[\Rightarrow \begin{bmatrix} \sigma_0 & 0 & 0 & \cdots & 0 \\ \sigma_1 & \sigma_0 & 0 & \cdots & 0 \\ \sigma_2 & \sigma_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \sigma_{n-1} & \cdots & \sigma_2 & \sigma_1 & \sigma_0 \end{bmatrix}^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_{n-1} \\ \sigma_n \end{bmatrix} = \begin{bmatrix} (-1)^1 S_1 \\ (-1)^2 S_2 \\ \vdots \\ (-1)^{n-1} S_{n-1} \\ (-1)^n S_n \end{bmatrix} \] 

(142)

If instead one is interested in characterizing Let

\[R_k = \left( \frac{(-1)^{n-k} S_{n-k}}{S_n} \right) \] 

(143)

we have

\[\Rightarrow \begin{bmatrix} \sigma_0 & 0 & 0 & \cdots & 0 \\ \sigma_1 & \sigma_0 & 0 & \cdots & 0 \\ \sigma_2 & \sigma_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \sigma_{n-1} & \cdots & \sigma_2 & \sigma_1 & \sigma_0 \end{bmatrix}^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_{n-1} \\ \sigma_n \end{bmatrix} = \begin{bmatrix} R_0 \\ R_{n-1} \\ \vdots \\ R_2 \\ R_1 \end{bmatrix} \] 

(144)

Let us first consider the example of the polynomial admitting as its roots the roots of unity.

Considering now the equation

\[\left( \sum_i R_i \cdot (-t)^i \right) \left( \sum_j (X \cdot t)^j \right) = \sum_i R_i \cdot (-t)^{i-1} \left( \frac{-t}{1-X \cdot t} \right) = 0 \] 

(145)

\[\Rightarrow \left( \sum_i R_i \cdot (-t)^i \right) \int X \left( \sum_j (x \cdot t)^j \right) dx = \sum_i R_i \cdot (-t)^{i-1} \int X \left( \frac{-t}{1-X \cdot t} \right) dx = 0 \] 

(146)
\[ \Rightarrow \left( \sum_i R_i \cdot (-t)^i \right) \left( \sum_j X_j^i \cdot (-t)^j \right)^{t-1} = \sum_i R_i \cdot (-t)^{i-1} \ln (1 - X \cdot t) = 0 \quad (147) \]

\[ \Rightarrow \left( \sum_i R_i \cdot (-t)^i \right) \left( \sum_j X_j \cdot (-t)^j \right) = \sum_i R_i \cdot (-t)^i \ln (1 - X \cdot t) = 0 \quad (148) \]

Let \( P(X) \) be some polynomial, we have

\[ P(X) = \prod_{1 \leq k \leq n} (X - X_k) \quad (149) \]

the newton symmetric polynomials are the polynomials defined by expression of the form

\[ \sigma_s (X_1, \cdots, X_n) = \sum_{1 \leq k \leq n} X_k^s \]

\[ \sigma_s (X_1, \cdots, X_n) = \int_{\pi}^{\pi} \prod_{1 \leq k \leq n} (1 - X_k \cdot e^{-iu}) e^{iu} du \quad (151) \]

It is rather straightforward to extend the preceding definition to complex number by simply computing the expression for complex values of the parameter \( s \). What might slightly less trivial is expressing a analytic continuation for the lagrange symmetric polynomials given by

\[ S_t (X_1, \cdots, X_n) = (-1)^t \sum_{X_1 < \cdots < X_n} \left( \prod_k X_k \right) \sum_{X_1 < \cdots < X_n} \left( \prod_k X_k \right) \]

These of course correspond to the coefficient of the expanded polynomial \( P(x) \). In order to compute \( S \) with complex values of the parameter \( t \), one should consider the following alternative definition for \( S \)

\[ S_t (X_1, \cdots, X_n, 0, 0, \cdots) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \prod_{1 \leq k \leq n} (e^{is} - X_k) \right) e^{-is(n-t)} \, ds \quad (153) \]

\[ S_t (x_1, \cdots, x_n, 0, 0, \cdots) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \prod_{1 \leq k \leq n} (1 - X_k \cdot e^{-is}) \right) e^{is} \, ds \quad (154) \]

It follows that for \( t > n \), \( S_t (x_1, \cdots, x_n, 0, 0, \cdots) = 0 \). The expression above allows us to prolonge analytic continuation of the elementary symmetric function from the integer values to complex values. We now consider the following sum
\[
\sum_t S_t = \frac{1}{2\pi} \int_{s=-\pi}^{s=\pi} \left( \prod_k (1 - X_k \cdot e^{-is}) \right) \sum_t e^{i s \cdot t} \, ds \quad (155)
\]

\[
\sum_t S_t = \int_{s=-\pi}^{s=\pi} \left( \prod_k (1 - X_k \cdot e^{-is}) \right) \frac{ds}{1 - e^{is}} \quad (156)
\]

From which it follows that

\[
\sin \frac{\pi}{\pi} = \int_{s=-\pi}^{s=\pi} \prod_k \left( 1 - e^{-i \frac{2s}{k^2}} \right) \frac{ds}{1 - e^{is}} \quad (157)
\]

Consider the expression below

\[
\left( \frac{\sin(\pi e^{-\frac{s}{2}})}{\pi e^{-\frac{s}{2}}} \right) = \prod_{n>0} \left( 1 - e^{-\frac{is}{n^2}} \right) \quad (158)
\]

\[
\Rightarrow S_t \left( 1, \frac{1}{2^2}, \cdots, \frac{1}{n^2}, \cdots \right) = \frac{1}{2\pi} \int_{s=-\pi}^{s=\pi} \left( \frac{\sin(\pi e^{-\frac{s}{2}})}{\pi e^{-\frac{s}{2}}} \right) e^{i s \cdot t} \, ds = \quad (159)
\]

\[
= \frac{1}{2\pi} \int_{s=-\pi}^{s=\pi} \prod_{n>0} \left( 1 - e^{-\frac{is}{n^2}} \right) e^{i s \cdot t} \, ds \quad (160)
\]

Let us discuss some further observation relating to the symmetric polynomials

\[
\frac{d}{dx} \sum_{i+j=n} (-1)^i S_i \cdot X^j = \left( \sum_{i+j=n} (-1)^i \cdot j \cdot S_i \cdot X^{j-1} \right) j > 0 \quad (161)
\]

\[
= \sum_t (X - X_t)^{-1} \left( \sum_{i+j=n} (-1)^i S_i \cdot X^j \right) \quad (162)
\]

\[
= \sum_t X_t \cdot \left( 1 - \frac{X}{X_t} \right)^{-1} \left( \sum_{i+j=n} (-1)^i S_i \cdot X^j \right) \quad (163)
\]

\[
\Rightarrow \sum_{i+j=n} (-1)^i \cdot j \cdot S_i \cdot X^{j-1} = - \sum_t \left( \sum_{i+j=n} (-1)^{n-j} S_{n-j} \cdot X_t^{1-i} \cdot X^{i+j} \right) \quad (164)
\]

\[
\Rightarrow \sum_{i+j=n} (-1)^i \cdot j \cdot S_i \cdot X^{j-1} = - \sum_{i+j=n} \left( (-1)^{n-j} \cdot S_{n-j} \cdot X^{i+j} \right) \sum_t X_t^{1-i} \quad (165)
\]
For the sine function we would have

\[
\sum_{i+j=n} (-1)^i \cdot j \cdot S_i \cdot X^{j-1} = - \sum_{i+j=n} \left( (-1)^{n-j} \cdot S_{n-j} \cdot X^{i+j} \right) \sum_{t} X_{i}^{t-i} \quad (166)
\]

\[
\sum_{i+j=n} (-1)^i \cdot j \cdot S_i \cdot X^{j-1} = - \sum_{i+j=n} \left( (-1)^{n-j} \cdot S_{n-j} \cdot \sigma_{1-i} \right) X^{i+j} \quad (167)
\]

For the sine function we would have

\[
\sum_{i+j=n} (-1)^i \cdot j \cdot S_i \cdot X^{j-1} = - \sum_{i+j=n} \left( (-1)^{n-j} \cdot S_{n-j} \cdot \sigma_{1-i} \right) X^{i+j} \quad (168)
\]

### 2.7 Revisiting the Riemann Zeta function

Jeff Lagarias showed that the Riemann hypothesis is equivalent to the following hypothesis is equivalent to the following elementary statement:

For all \( n \geq 1 \)

\[
\sum_{d|n} d \leq H_n + \exp \left( H_n \right) \log \left( H_n \right) \quad (169)
\]

Some notation and a function

\[
n = \vec{p}^\vec{a} \equiv \prod_{1 \leq k \leq |n|} p_k^{\alpha_k} \quad (170)
\]

Let

\[
H_n (x) = \sum_{1 \leq k \leq n} \exp \left( \frac{x}{k} \right) \quad (171)
\]

\[
\left( \sum_{0 \leq a \leq \vec{a}} \exp \left( x \cdot \vec{p}^a \right) \right) \leq \log \left\{ (H_n(x))^{\exp \{ H_n(x) \} } \right\} \quad (172)
\]

From which we would want to prove that

\[
\exp \left\{ \sum_{0 \leq a \leq \vec{a}} \exp \left( x \cdot \vec{p}^a \right) \right\} \leq (H_n(x))^{\exp \{ H_n(x) \} } \quad (173)
\]

\[
\exp \left\{ \sum_{0 \leq a \leq \vec{a}} \exp \left( x \cdot \vec{p}^a \right) \right\} \leq \quad (174)
\]

\[
\left( 1 + (1 - \gamma) x - \left( \frac{\pi^2}{6} \right) x^2 - \sinh(x) + 2 \log \left( \Gamma \left( 1 - x \right) \right) \right)^{\exp \{ (1 - \gamma) x \} \exp \left\{ - \left( \frac{\pi^2}{6} \right) x^2 \right\} \exp \{ - \sinh(x) \} \Gamma^a \left( 1 - x \right)} \quad (175)
\]
\[
1 + (1 - \gamma)x - \left(\frac{\pi^2}{6}\right)x^2 - \sinh(x) + 2\log(\Gamma(1-x)) \quad (176)
\]

\[
\frac{1}{\Gamma^2(1-x)} \sum_{\vec{a} \leq \vec{a}} \exp \left(-1 - (1 - \gamma - \vec{p})x + \left(\frac{\pi^2}{6}\right)x^2 - \sinh(x)\right) \leq \log(\log(n)) \quad (177)
\]

Riemann starting point for his seminal paper was Euler’s identity

\[
\prod_k \left(\sum_{u} p_u^{-su}\right) = \sum_{n \in \mathbb{N}} n^{-s} \quad (178)
\]

The inspiration for this section came from one of Johann Bernoulli 1697 result which is that

\[
\int_0^1 x^{-x}dx = \sum_{n \in \mathbb{N}^*} n^{-n}. \quad (179)
\]

Here is how we prove it

\[
\int_0^1 x^{-x}dx = \int_0^1 e^{-x\ln x}dx = \int_0^1 \sum_{r \in \mathbb{N}} \frac{(-1)^r}{r!} (\ln x)^r dx
\]

\[
= \sum_{r \in \mathbb{N}} \frac{(-1)^r}{r!} \int_0^1 (\ln x)^r dx
\]

\[
= 1 + \sum_{r \in \mathbb{N}^*} \frac{(-1)^r}{r!} \int_0^1 x^r \ln^r(x)dx
\]

focusing on the integral we note that

\[
\int_0^1 x^r \ln^r(x)dx = \left[\frac{x^{r+1}}{r+1} \ln^r(x)\right]_0^1 = \frac{r}{r+1} \int_0^1 \frac{x^{r+1}}{x} \ln^{r-1}(x)dx
\]

\[
= -\frac{r}{r+1} \int_0^1 x^r \ln^{r-1}(x)dx
\]

\[
= (-1)^r \frac{r!}{(r+1)^{r+1}}
\]

and so

\[
26
\]
\[ \int_0^1 x^{-x} dx = 1 + \sum_{r \in \mathbb{N}^*} \frac{(-1)^r}{r!} \left( -1 \right)^r \frac{r!}{(r + 1)^{r+1}} \]  

(187)

\[ \int_0^1 x^{-x} dx = 1 + \sum_{r \in \mathbb{N}^*} (r + 1)^{-(r+1)} \]  

(188)

The above expression is of interest and we may make a few observations about it summarized as follows. Consider the functions

\[ \int_0^1 x^{-x} dx = \left[ \int_0^1 \left( \frac{d}{dt} \right)^x e^t \right]_{t=0} = \sum_{n \in \mathbb{N}^*} n^{-n} \]  

(189)

\[ \int_0^1 x^{-x} dx = \left[ \sum_{n \in \mathbb{N}^*} \left( \frac{d}{dt} \right)^n e^t \right]_{t=0} = \sum_{n \in \mathbb{N}^*} n^{-n} \]  

(190)

Let us furthermore introduce the following functions

\[ q_k(x) = \left( \sum_{r \in \mathbb{N}} \frac{x^r}{(k \cdot r)!} \right) = \left( \sum_{k \mid r} \frac{x^r}{r!} \right) \]  

(191)

for

\[ \phi(r) = \begin{cases} 1 & \text{if } r \equiv 0 \text{ mod } (k) \\ 0 & \text{else} \end{cases} \]  

(192)

equivalently we write.

\[ q_k(x) = \left( \sum_{r \in \mathbb{N}} \frac{\phi(r)}{r!} x^r \right) \]  

(193)

\[ \int_0^1 q_k(-x^k \cdot \ln^k(x)) dx = \sum_{r \in \mathbb{N}} (-1)^{k \cdot r} \frac{1}{(k \cdot r)!} \int_0^1 x^k \cdot \ln^k(x) dx \]  

(194)

We now focus on the expression inside the integral.

\[ \int_0^1 x^{k \cdot r} \cdot \ln^{k \cdot r}(x) dx = \left[ x^{k \cdot r+1} \cdot \ln^{k \cdot r}(x) \right] - \frac{k \cdot r}{k \cdot r + 1} \int_0^1 x^{k \cdot r+1} \cdot \ln^{k \cdot r-1}(x) dx \]  

(195)

\[ = \cdots (-1)^{k \cdot r} \frac{(k \cdot r)!}{(k \cdot r + 1)^{k \cdot r}} \int_0^1 x^{k \cdot r} dx \]  

(196)
\[
(-1)^{k \cdot r} \frac{(k \cdot r)!}{(k \cdot r + 1)^{k \cdot r + 1}}
\]

(197)

and so

\[
\int_0^1 q_k \left( -x^k \cdot \ln^k(x) \right) \, dx = 1 + \sum_{r \in \mathbb{N}} \frac{(-1)^{k \cdot r}}{(k \cdot r)!} \frac{(k \cdot r)!}{(k \cdot r + 1)^{k \cdot r + 1}}
\]

(198)

\[
\int_0^1 q_k \left( -x^k \cdot \ln^k(x) \right) \, dx = 1 + \sum_{r \in \mathbb{N}} \frac{1}{(k \cdot r + 1)^{k \cdot r + 1}}
\]

(199)

2.8 Some problems in Number theory

2.8.1 The Perfect Number Conjecture

The perfect number conjecture states that all perfect numbers are even

\[
2 \left( \prod_{1 \leq k \leq n} p_k^{\alpha_k} \right) = \prod_{1 \leq k \leq n} \left( \frac{1 - p_k^{1+\alpha_k}}{1 - p_k} \right)
\]

(200)

Let us now briefly discuss a combinatorial interpretation for perfection. We recall that

\[
[n]_q = \left( \frac{1 - q^n}{1 - q} \right)
\]

(201)

It follows that the condition for a number to be perfect is the following

\[
\left( \prod_{1 \leq k \leq m} [1 + n_k]_{p_k} \right) = 2 \left( \prod_{1 \leq k \leq m} p_k^{n_k} \right)
\]

(202)

Here are some observations on even perfect numbers

\[
\left( \frac{(x - 2)! \cdot (x^m - x)!}{(x^m - 2)!} \right) \left( \frac{x^m}{x} \right) = x^{m-1} \left( \frac{x^m - 1}{x - 1} \right)
\]

(203)

\[
\left( \frac{x! \cdot (x^m - x)!}{2 \cdot (x^m - 2)!} \right) \left( \frac{x^m}{x} \right) = \frac{x^m (x^m - 1)}{2}
\]

(204)

For even perfect number the following identity holds

\[
[2]_{(2m-1)} \times [m]_2 = \left( \frac{2}{1} \right) \times \left( \frac{2^m}{2} \right)
\]

(205)

Find solutions to the equation
\[ x^m - [x]_{(x^{m-1})} \times [m]_x = 0 \]  
(206)

Let us summarize the problem for

\[ [x]_y \equiv \left( \frac{1 - y^x}{1 - y} \right) \]  
(207)

Find values of \( m \) and \( x \) which satisfies the following equality

\[ x \times \left( \frac{x^m}{x} \right) - [x]_{(x^{m-1})} \times [m]_x = 0 \]  
(208)

### 2.8.2 Perfect Numbers Entropy and Trigonometry

Reading Alfred Renyi master piece titled : On Measure of Entropy and Information I learned the following.

Let \( P = (p_k)_{1 \leq k \leq n} \) be a finite discrete probability distribution. The amount of uncertainty concerning the outcome of an experiment the result of which have the probabilities \( p_k \) for \( 1 \leq k \leq n \), is called the entropy of the distribution \( P \) and is usually measured by the quantity \( H[P] = H(p_1, \cdots, p_n) \) introduced by Shannon and defined by

\[
H(p_1, \cdots, p_n) = \sum_{1 \leq k \leq n} p_k \cdot \log \left( \frac{1}{p_k} \right) 
\]  
(209)

Different sets of postulates have been proposed, which characterize the expression above

1. \( H \) is symmetric in its variables
2. \( H(p, 1 - p) \) is a continuous function of \( p \) for \( 0 \leq p \leq 1 \).
3. \( H \left( \frac{1}{2}, \frac{1}{2} \right) = 1 \).
4. \( H(t \cdot p_1, (1 - t) \cdot p_1, p_2, \cdots, p_n) = H(p_1, p_2, \cdots, p_n) + p_1 \cdot H(t, (1 - t)) \) for any distribution and for \( 0 \leq t \leq 1 \).

The paper claims that these postulates completely determine the expression of the entropy proposed by Shannon. I find this result quite surprising even though I will not discuss the proof of this assertion. I will mention that the last point of the discussion prompts me to revisit the trigonometric description of discrete probability distribution.

Consider \( P = (p_1, p_2) \) we know that \( 1 = p_1 + p_2 \) so for

\[
\arctan \left( \frac{\sqrt{p_2}}{\sqrt{p_1}} \right) = \theta 
\]  
(210)

\[
\begin{aligned}
\cos^2(\theta) &= p_1 \\
\sin^2(\theta) &= p_2 
\end{aligned} 
\]  
(211)
\begin{align*}
p_1 &= \cos^2(\theta_1) \quad (212) \\
1 < m < n \quad p_m &= \left(\prod_{k=1}^{m-1} \sin^2(\theta_k)\right) \cos^2(\theta_m) \quad (213) \\
p_n &= \prod_{k=1}^{n} \sin^2(\theta_k) \quad (214)
\end{align*}

for some \( \vec{\theta} \). the 4th postulate is translated into

\[
\mathcal{H} \left( \cos^2(\theta) \cdot p_1, \sin^2(\theta) \cdot p_1, p_2, \cdots p_n \right) = \mathcal{H} (p_1, p_1, p_2, \cdots p_n) + p_1 \cdot \mathcal{H} \left( \cos^2(\theta), \sin^2(\theta) \right) \quad (215)
\]

this has significant implication

\[
\mathcal{H} \left( \cos^2(\theta_1), \sin^2(\theta_1) \cdot \cos^2(\theta_2), \sin^2(\theta_1) \cdot \sin^2(\theta_2) \right) = \\
\mathcal{H} \left( \cos^2(\theta_1), \sin^2(\theta_1) \right) + \sin^2(\theta_1) \cdot \mathcal{H} \left( \cos^2(\theta_2), \sin^2(\theta_2) \right) \quad (216)
\]

So given the trigonometric representation of a discrete probability distribution we can write

\[
\mathcal{H} [P] = 
\]

\[
\mathcal{H} \left( \cos^2(\theta_1), \sin^2(\theta_1) \right) + \sin^2(\theta_1) \cdot \left( \cdots + \sin^2(\theta_{n-1}) \cdot \mathcal{H} \left( \cos^2(\theta_{n-1}), \sin^2(\theta_{n-1}) \right) \right) \quad (217)
\]

The Shannon entropy can be expressed as weighed combination of elementary probability \( \mathcal{H} \left( \cos^2(\theta_k), \sin^2(\theta_k) \right) \). Here is the sequence that can be though of as generating \( \mathcal{H} [P] \).

\[
h_0 = \mathcal{H} \left( \cos^2(\theta_1), \sin^2(\theta_1) \right) \quad (218)
\]

\[
h_1 = \mathcal{H} \left( \cos^2(\theta_1), \sin^2(\theta_1) \right) + \sin^2(\theta_1) \cdot \mathcal{H} \left( \cos^2(\theta_2), \sin^2(\theta_2) \right) \quad (219)
\]

\[
h_1 = h_0 + \sin^2(\theta_1) \cdot \mathcal{H} \left( \cos^2(\theta_2), \sin^2(\theta_2) \right) \quad (220)
\]

and son on ...

The iteration can be described by the following change of variable in the in \( h_{k-1} \)

\[
\mathcal{H} \left( \cos^2(\theta_{k-1}), \sin^2(\theta_{k-1}) \right) \rightarrow \left[ \mathcal{H} \left( \cos^2(\theta_{k-1}), \sin^2(\theta_{k-1}) \right) + \sin^2(\theta_{k-1}) \cdot \mathcal{H} \left( \cos^2(\theta_k), \sin^2(\theta_k) \right) \right] 
\quad (221)
\]
Let us now discuss the relationship with perfect numbers. Let the prime expansion of the perfect number \( m \) be

\[
m = \prod_{1 \leq k \leq n} p_k^{a_k}
\]

we have

\[
\left( \sum_{d|m, \; d > 1} \frac{1}{d} \right) = 1 \quad (223)
\]

So the inverse of the divisors of a perfect number represent a probability distribution, what is more is that we can say something about the entropy of the distribution as follows

\[
\Rightarrow \left( \sum_{d|m, \; d > 1} \frac{1}{d} \log (d) \right) + \left( \sum_{d|m, \; d > 1} \frac{1}{d} \log \left( \frac{m}{d} \right) \right) = \log (m) \quad (224)
\]

\[
\Rightarrow m \left( \sum_{d|m, \; d > 1} \frac{1}{d} \log (d) \right) + \left( \sum_{d|m, \; d > 1} \left( \frac{m}{d} \right) \log \left( \frac{m}{d} \right) \right) = m \log (m) \quad (225)
\]

\[
\Rightarrow -m \left( \sum_{d|m, \; d > 1} \frac{1}{d} \log (d) \right) + \left( \sum_{d|m, \; d > 1} \left( \frac{m}{d} \right) \log \left( \frac{1}{\left( \frac{m}{d} \right)} \right) \right) = m \log \left( \frac{1}{m} \right) \quad (226)
\]

\[
\Rightarrow -m \left( \sum_{d|m, \; d > 1} \left( \frac{1}{d} \right) \log (d^{-1}) \right) + \left( \sum_{d|m, \; d > 1} d \cdot \log (d^{-1}) \right) = m \log \left( \frac{1}{m} \right) \quad (227)
\]
What can I say in general about beside that these are exspetation of particular function

\[ H_s (d_1, \cdots, d_{\Pi_\alpha (1+\alpha)}) = \sum_{1 \leq k \leq \Pi_\alpha (1+\alpha)} d_k^s \cdot \log \left( \frac{1}{d_k^s} \right) = E \left[ \log \left\{ x^{-sx^{-1}} \right\} \right] \]  
(228)

\[ = E \left[ \log \left\{ x^{-\frac{1}{sx}} \right\} \right] \]  
(229)

I shall ask Dave In any case we have obtained that for perfect number \( m \)

\[ H_{-1} (d_1, \cdots, d_{\Pi_\alpha (1+\alpha)}) - m \cdot H (d_1, \cdots, d_{\Pi_\alpha (1+\alpha)}) = m \log \left( \frac{1}{m} \right) \]  
(230)

I shall investigate subsequently implications of the trigonometric expansion of perfect numbers

3 Tower Arithmetics

Definition

A tower expansions (or simply a tower) over the variable \( \{x_1, x_2, \cdots, x_n\} \) is finite product of finitely iterated exponentiation over the variables. Let \( X \equiv (x_1, x_2, \cdots, x_n) \) denote the vector of variables. The set of tower expansions over the entries of \( X \) is denoted \( T(X) \).

Example

Let \( X = (x, y) \) denotes the vector of variables. the example features the following 3 towers :

\[ x, \left( x^y \right), \left( x^{x^y} y^x, y^{x^y} y^y \right) \]  
(231)

The height of a tower corresponds to the maximum number of iterated exponentiations occuring in the tower. The base of the tower refers to the bottom level of the tower expansion. Furthermore if the base of the tower is made of a single variable we say that the tower expansion is made of a single pillar.

Theorem : Every positive integer greater than 1 can be written uniquely as a product of primes with the primes factors written in non decreasing order.

Corollary : Every positive integer greater than 1 can be written uniquely as a tower expansion over the primes. The pillars of the subtowers written in increasing order.
3.1 Formal Tower Series

A Formal Tower Series is a series made of linear combinations (of not necessarily finite) set of distinct towers. The coefficient chosen from some field \( \mathbb{F} \) usually \( \{0, 1\} \). The set of Formal Tower Series is denoted \( \mathbb{F}[\mathcal{T}(X)] \) where

\[
X \equiv (x_1, x_2, \ldots, x_k, \ldots)
\]

denotes the vector of variables. When the formal tower series is made of linear combination of finitely many towers we refer to such expression as superpolynomials and the terms in the expression are called supermonomials.

3.2 Revisiting Euler’s Identity

Let

\[
X \equiv (x_1, \ldots, x_k, \ldots)
\]

denotes an infinite dimensional vector of variables. As a result of the fundamental theorem of arithmetics the integers induce a bijective correspondence between \( \mathcal{T}(X) \) and \( \mathbb{N} \setminus \{0, 1\} \). Let us illustrate here the correspondence as follows

\[
\begin{align*}
T_2(X) &= x_1 \\
T_3(X) &= x_2 \\
T_4(X) &= x_1^2 \\
T_5(X) &= x_3 \\
T_6(X) &= x_1 \cdot x_2 \\
T_7(X) &= x_4 \\
T_8(X) &= x_1^2 \cdot x_2 \\
T_9(X) &= x_5 \\
T_{10}(X) &= x_1 \cdot x_3 \\
T_{11}(X) &= x_5
\end{align*}
\]

We now discuss a binary operator \( R(\cdot, \cdot) \) which will be of significant importance throughout this text.

\[
R : \{T_k(X)\}_{2 \leq k \leq \infty} \times \mathcal{T}(X) \rightarrow \mathcal{T}(X)
\]

\[
R \left( x_i, \sum_{n \in \mathbb{N} \setminus \{0, 1\}} a_n \cdot T_n(X) \right) = \sum_{n \in \mathbb{N} \setminus \{0, 1\}} a_n \cdot x_i \cdot T_n(X)
\]

We recall the Euler identity is

\[
\prod_k (1 - p_k^{-s})^{-1} = \sum_n n^{-s}
\]

may be thought of as an invariance principle. The benefits of introducing Formal Tower Series is the fact that we may validate expressions of the form
\[
\left( \prod_{1 \leq k \leq \infty} (1 - p_k)^{-1} \right) = \sum_{1 \leq k \leq \infty} k
\]  

(238)

while obviating the need to be concerned about convergence. We now discuss the invariance principle.

\[
\prod_{1 \leq k \leq \infty} \left( 1 + R \left( x_k, \sum_{n \in \mathbb{N} \setminus \{0,1\}} T_n (X) \right) \right) = \sum_{n \in \mathbb{N} \setminus \{0,1\}} T_n (X)
\]  

(239)

here is how we start off with the entries of \( X \) to induce the \( T_n (X) \)s. Let \( g (x) \) be the function defined by

\[
g (x) = x + x^x + x^{x^x} + x^{x^{x^x}} + \cdots
\]  

(240)

We introduce a sequence of Tower Series defined as follows

\[
G_0 (X) = \prod_{1 \leq k \leq |X|} g (x_k)
\]  

(241)

where \(|X|\) refers to the dimensions of the vector of variables.

\[
G_{n+1} (X) = \prod_{1 \leq k \leq |X|} (1 + R (x_k, G_n (X)))
\]  

(242)

Euler’s identity depicts an invariance principle for the set of primes.

\[
\lim_{n \to \infty} \{G_n (X)\} = 1 + \sum_{k \in \mathbb{N} \setminus \{0,1\}} T_k (X)
\]  

(243)

The particular ordering over the integers induces an ordering over the supermonomials , from which the algebra of integers described by

\[
d (T_m (X), T_p (X)) = d (1, T_n (X)) \Leftrightarrow T_m (X) + T_n (X) = T_p (X)
\]  

(244)

where \(d (\cdot, \cdot)\) refers to the distance operator and

\[
T_m (X) < T_p (X)
\]  

(245)

Considering the sequence define for the finite dimensional vector \( P \equiv (p_1, \cdots, p_n) \)

\[
G_0 (P) = 1 + \prod_{1 \leq k \leq |P|} g (p_k)
\]  

(246)

\[
G_{m+1} (P) = \prod_{1 \leq k \leq |P|} (1 + R (p_k, G_m (P)))
\]  

(247)
\[
\lim_{m \to \infty} \{G_m(P)\} = 1 + \sum_{k \in \mathbb{N}\{0,1\}} a_k \cdot T_k(P)
\]  

where \(a_k \in \{0,1\}\)

**Theorem**: For every finite set of primes \(P = (p_k)_{1 \leq k \leq n}\) and for

\[
\lim_{m \to \infty} \{G_m(P)\} = 1 + \sum_{k \in \mathbb{N}\{0,1\}} a_k \cdot T_k(P)
\]  

the series

\[
S(P) \equiv \left(1 + \sum_{k \in \mathbb{N}\{0,1\}} a_k \cdot (T_k(P))^{-1}\right) < \infty
\]  

converges.

The proof of the convergence follows from the product of geometric progressions.

\[
\left(1 + \sum_{k \in \mathbb{N}\{0,1\}} a_k \cdot (T_k(P))^{-1}\right) < \left(\prod_{1 \leq k \leq n} (1 - p_k^{-1})\right) < \infty
\]

3.3 Getting hold of the rationals.

We recall that

\[
g(x) = x + x^x + x^{x^x} + x^{x^{x^x}} + \cdots
\]

Let

\[
P \equiv (p_1, \cdots, p_k, \cdots)
\]

denote the vectors of the primes.

\[
\lim_{n \to \infty} \left\{H_n(P) = \prod_{1 \leq k \leq |P|} (R(p_k^{-1}, G_n(P)) + 1 + R(p_k, G_n(P)))\right\}
\]

The terms in the sums are in bijective correspondence with the element on \(\mathbb{Q}\)

3.4 Tower Sieve Algorithm.

Let us discuss a Sieving algorithm following from tower arithmetics. Let \(P \equiv (p_1, \cdots, p_n)\) denote the set of primes that are less or equal to \(2^t\), we also define \(g_s\)

\[
g_s(x) = x + (x^x) + (x^{x^x}) + (x^{x^{x^x}}) + \cdots + \left(x^{\cdots x}\right) \text{ height } = s
\]
the sieve will determine the primes in the range \([2^t, 2^{t+1}]\). The algorithm consists in computing the following recursion

\[
G_0(P) = \prod_{1 \leq k \leq |P|} g_{s_k}(p_k) \tag{256}
\]

\(s_k\) denoted the maximum height the tower of the form \(p_k \cdot \ldots \cdot p_k\) such that \(p_k \cdot \ldots \cdot p_k \leq 2^{t+1}\).

\[
G_{m+1}(P) = \prod_{1 \leq k \leq |P|} (1 + R(p_k, G_m(P))) \tag{257}
\]

we stop the recursion at \(G_u(P)\) if the expression

\[
G_{u+1}(P) - G_u(P) \tag{258}
\]

is made of terms all greater than \(2^{t+1}\). Observing that \(G_u(P)\) determines the integers less than \(2^{t+1}\) with the exception of the primes in the range \([2^t, 2^{t+1}]\). Which are identified by locating gaps of step size equal 2.

**Example:**

First let us determine the primes in the interval \([2^2, 2^2] \equiv [p_1, p_1^{p_1}]\)

\[
P \equiv (p_1) \tag{259}
\]

denotes the set of primes less or equal to \(2 = p_1\).

\[
g_2(p_1) = p_1 + p_1^{p_1} \tag{260}
\]

\[
G_0(P) = 1 + p_1 + p_1^{p_1} \tag{261}
\]

\[
G_1(P) = 1 + R(p_1, 1 + p_1 + p_1^{p_1}) \tag{262}
\]

\[
= 1 + p_1 + p_1^{p_1} + p_1^{p_1^{p_1}} \tag{263}
\]

We note that

\[
G_1(P) - G_0(P) = p_1^{p_1^{p_1}} \tag{264}
\]

which is a tower outside the range. We therefore stop at \(G_0(P)\), the terms present in the sums presented in order are

\[
\{1, p_1, p_1^{p_1}\} \tag{265}
\]

we see from the list that there is only one gap of size 2 and it is between \(p_1\) and \(p_1^{p_1}\). Which determines the relation

\[
p_1 < p_2 < p_1^{p_1}. \tag{266}
\]
Thereby yielding the completed super polynomial expression

$$1 + p_1 + p_2 + p_1^{p_2}$$  \hspace{1cm} (267)

Building on this we may try to determine the numbers the primes in the range $[2^2, 2^3] \equiv [p_1^{p_2}, p_1^{p_2}]$. In this particular instance

$$P \equiv (p_1, p_2)$$  \hspace{1cm} (268)

$$G_0 (P) = 1 + p_1 + p_2 + p_1^{p_3}$$  \hspace{1cm} (269)

$$G_1 (P) = (1 + R(p_1, 1 + p_1 + p_2)) (1 + R(p_2, 1))$$  \hspace{1cm} (270)

$$= (1 + p_1 + p_1^{p_3} + p_1^{p_2}) (1 + p_2)$$  \hspace{1cm} (271)

$$= 1 + p_1 + p_2 + p_1^{p_3} + p_2 p_1 + p_1^{p_2} + p_2 p_1^{p_1} + p_2 p_1^{p_2}$$  \hspace{1cm} (272)

The last two terms being larger than $p_1^{p_2}$ we may drop them from the sum to obtain

$$G_1 (P) = 1 + p_1 + p_2 + p_1^{p_3} + p_2 p_1 + p_1^{p_2}$$  \hspace{1cm} (273)

from the terms present we see that there are exactly two gaps of step size equal to 2 located between $p_1^{p_3}$ and $p_2 p_1$ for the first gap and $p_2 p_1$ and $p_1^{p_2}$ for the second gap. Which determines the relations.

$$p_1^{p_3} < p_3 < p_2 p_1$$  \hspace{1cm} (274)

and

$$p_2 p_1 < p_4 < p_1^{p_2}$$  \hspace{1cm} (275)

to yield the completed superpolynomial

$$1 + p_1 + p_2 + p_1^{p_3} + p_3 + p_2 \cdot p_1 + p_4 + p_1^{p_2}$$  \hspace{1cm} (276)

For the next iteration we will start with

$$G_0 (P) = 1 + p_1 + p_2 + p_1^{p_3} + p_3 + p_2 \cdot p_1 + p_4 + p_1^{p_2}$$  \hspace{1cm} (277)

We presented above an iterative algorithm which generates primes $p_k > 2$. 

37
3.4.1 Recovering the Ordering.

In the expressions above we assumed we knew the values of the primes below $2^t$. We now describe an algorithm for recovering the ordering for the first few towers. The ordering of the first few towers from the binary expansion of the primes and the equivalence class associated with each one of the binary expansions discussed in \[\]. We have discussed how to generate the superpolynomials out of a ordered finite set of variables stored in finite dimensional vector $X$. We define for the purpose

$$g_s(x_{i,t}) = x_{i,t} + \left(\frac{x_{i,t}}{x_{i,t}}\right) + \cdots + \left(\frac{x_{i,t}}{x_{i,t}}\right) \text{ height } = s$$ (278)

$$G_0(X) = \prod_{1 \leq i \leq |X|} g_{s_i}(x_{i,0})$$ (279)

We define the following iteration

$$G_m(P) = \prod_{1 \leq i \leq |X|} (1 + R(x_{i,m}, G_{m-1}(X)))$$ (280)

In order to illustrate consider the following product

$$(1 + R(x_{1,2}, x_{2,1}^3x_{1,1} + x_{1,1} + x_{2,1} + 1))(1 + R(x_{2,2}, x_{1,1} + 1))$$ (281)

which results in the following list of terms

$$\{1, x, x^2, x^{2x_{1,1}}, x^{2x_{2,1}}, (x + 1), x \cdot (x + 1), x^2 (x + 1), x^{x_{2,1}} (x + 1), x^{2x_{2,1}} (x + 1),$$

$$(x + 1)^2, x \cdot (x + 1)^2, x^2 \cdot (x + 1)^2, x^{x_{2,1}} (x + 1)^2, x^{2x_{2,1}} (x + 1)^2\}$$ (284)

$$x_{1,1} \rightarrow 2$$ (282)

which

$$\{1, x, x^2, x^{x_{1,1}}, x^{2x_{2,1}}, (x + 1), x \cdot (x + 1), x^2 (x + 1), x^{x_{2,1}} (x + 1), x^{2x_{2,1}} (x + 1),$$

$$(x + 1)^2, x \cdot (x + 1)^2, x^2 \cdot (x + 1)^2, x^{x_{2,1}} (x + 1)^2, x^{2x_{2,1}} (x + 1)^2\}$$ (284)

$$x_{2,2} \rightarrow (x + 1)$$ (285)

Form which we deduce that

$$x_{2,1} \rightarrow 3$$ (286)

and end up with

$$\{1, x, x^2, x^3, x^{2x_{1,1}}, (x + 1), x \cdot (x + 1), x^2 (x + 1), x^3 (x + 1), x^{2x_{1,1}} (x + 1),$$

$$(x + 1)^2, x \cdot (x + 1)^2, x^2 \cdot (x + 1)^2, x^{x_{2,1}} (x + 1)^2, x^{2x_{2,1}} (x + 1)^2\}$$ (284)
\[(x + 1)^2, x \cdot (x + 1)^2, x^2 \cdot (x + 1)^2, x^3 \cdot (x + 1)^2, x^2 \cdot (x + 1)^2\] (287)

Which can determines the binary expansions for the first few towers associated with the first number

\[\left\{ 1, x, (x + 1), x^2, x \cdot (x + 1), x^3, (x + 1)^2, x^2 \cdot (x + 1)^2, x^3 \cdot (x + 1)^2, x \cdot (x + 1)^2 \right\} \] (288)

### 3.4.2 Addition Algorithm.

We assume that numbers are given in their prime tower representation in terms of binary expansion of the primes, the basis for the addition algorithm is recursion algorithm expressed by

\[
\begin{align*}
n + m &= (\sqrt{n} + \sqrt{m})^2 - 2\sqrt{n} \cdot \sqrt{m} \\
n - m &= (\sqrt{n} - \sqrt{m}) \cdot (\sqrt{n} + \sqrt{m})
\end{align*}
\] (289)

#### 3.5 Sum Product relations

Let us suppose the following relation given and known to be true.

\[
\sum_{0 \leq k \leq \infty} f(k, s) = \prod_{1 \leq k \leq \infty} (1 - g(k, s))
\] (290)

Let us consider the following reformulations

\[
\lim_{n \to \infty} \left\{ \sum_{0 \leq k \leq n} f(k, s) \right\} = \lim_{n \to \infty} \left\{ \prod_{1 \leq k \leq n} (1 - g(k, s)) \right\}
\] (291)

Borrowing from the Riemann integral framework we may rewrite th expression above as follows

\[
\lim_{n \to \infty} \left\{ n^{\frac{1}{n}} \left( \sum_{0 \leq k \leq n} f\left(\frac{k \cdot n}{n}, s \right) \right)^{\frac{1}{n}} \right\} = \lim_{n \to \infty} \left\{ \exp \left[ \sum_{1 \leq k \leq n} \ln \left(1 - g\left(\frac{k}{n} \cdot n, s \right) \right) \frac{1}{n}\right] \right\}
\] (292)

\[
\lim_{n \to \infty} \left\{ \left( \int_{0}^{1} f(t \cdot n, s) \, dt \right)^{\frac{1}{n}} \right\} = \lim_{n \to \infty} \left\{ \exp \left[ \int_{0}^{1} \ln \left(1 - g(t \cdot n, s) \right) \, dt \right] \right\}
\] (293)
\[
\lim_{n \to \infty} \left\{ \ln \left[ \left( \int_0^1 f(t \cdot n, s) \, dt \right)^{\frac{1}{n}} \right] \right\} = \lim_{n \to \infty} \left\{ \int_0^1 \ln \left( 1 - g(t \cdot n, s) \right) \, dt \right\} \quad (294)
\]

We now discuss some identities which result from the sum product relation

1. Euler’s Identity

\[
\lim_{n \to \infty} \left\{ \sum_{1 \leq k \leq n} k^{-s} \right\} = \lim_{m \to \infty} \left\{ \prod_{1 \leq k \leq m} \left( 1 - p^{-s}(k) \right) \right\} \quad (295)
\]

\[
\lim_{n \to \infty} \left\{ \ln \left[ \left( n^{-s} \int_0^1 t^{-s} \, dt \right)^{\frac{1}{n}} \right] \right\} = \lim_{m \to \infty} \left\{ \int_0^1 \ln \left( 1 - p^{-s}(t \cdot m) \right) \, dt \right\} \quad (296)
\]

\[
0 = \lim_{m \to \infty} \left\{ \int_0^1 \ln \left( 1 - p^{-s}(t \cdot m) \right) \, dt \right\} \quad (297)
\]

2. The Sine Function/General polynomial trick: we recall that

\[
\left( \sum_{0 \leq j \leq n} R_j \cdot X^j \right) = \prod_{1 \leq k \leq n} \left( 1 - \frac{X}{X_k} \right) \quad (298)
\]

for convinience I identified respectively \( X_k \) with the function \( X(k) \) and \( R_k \) with the function \( R(k) \)

\[
n \left( \sum_{0 \leq k \leq n} R(k) \cdot X \right) \cdot e^{\frac{1}{n} \ln(X^n)} = \exp \left\{ \sum_{1 \leq k \leq n} \ln \left( 1 - \frac{X}{X_k} \right) \right\} \quad (299)
\]

\[
\Rightarrow \left( \int_0^1 R(t \cdot n) \cdot e^{t \ln(X^n)} \, dt \right)^{\frac{1}{n}} = \exp \left\{ \int_0^1 \ln \left( 1 - \frac{X}{X(t \cdot n)} \right) \, dt \right\} \quad (300)
\]

In particular we have

\[
(1 - X^n)^{\frac{1}{n}} = \prod_{1 \leq k \leq n} \left( 1 - \frac{X}{e^{\frac{1}{n} \ln(X^n)}} \right)^{\frac{1}{n}} \quad (301)
\]

\[
\lim_{n \to \infty} \left\{ (1 - X^n)^{\frac{1}{n}} \right\} = \lim_{n \to \infty} \left\{ \left( \frac{1 - X^n}{1 - X} \right)^{\frac{1}{n}} \right\} = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \ln \left( 1 - X \cdot e^{-it} \right) \, dt \right\} \quad (302)
\]
\[
\lim_{n \to \infty} \left\{ n^{\frac{1}{n}} \left( \sum_{0 \leq k \leq n-1} (X^n)^{\frac{k}{n}} \right)^{\frac{1}{n}} \right\} = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \ln \left( 1 - X \cdot e^{-it} \right) \, dt \right\}
\]

(303)

\[
\lim_{n \to \infty} \left\{ \left( \int_0^{1} X^n \, dt \right)^{\frac{1}{n}} \right\} = \lim_{n \to \infty} \left\{ \left( \frac{X^n - 1}{n \log(X)} \right)^{\frac{1}{n}} = \left( \frac{X^n - 1}{\log(X)} \right)^{\frac{1}{n}} \right\} = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \ln \left( 1 - X \cdot e^{-it} \right) \, dt \right\}
\]

(304)

\[
= \begin{cases} 
1 & \text{if } |X| \leq 1 \\
X & \text{if } |X| > 1 
\end{cases}
\]

(305)

Let us consider the example of the sine function

\[
\frac{\sin(X)}{X} = \sum_{k=0}^{\infty} \frac{(-X^2)^k}{\Gamma(2(k + 1))} = \prod_{k=1}^{\infty} \left( 1 - \frac{X^2}{\pi^2 k^2} \right)
\]

(306)

\[
\lim_{n \to \infty} \left( \int_0^{1} e^{n \cdot t(i\pi + 2\ln(X))} \frac{1}{\Gamma(2(t \cdot n + 1))} \, dt \right)^{\frac{1}{n}} = \lim_{n \to \infty} \exp \left\{ \int_0^{1} \ln \left( 1 - \frac{X^2}{\pi^2 (t \cdot n)^2} \right) \, dt \right\}
\]

(307)

I find that assuming that \( \Re \left( \left( \frac{X}{n} \right)^2 \right) < 0 \) we get

\[
\lim_{n \to \infty} \left( \int_0^{1} e^{n \cdot t(i\pi + 2\ln(X))} \frac{1}{\Gamma(2(t \cdot n + 1))} \, dt \right)^{\frac{1}{n}} = \lim_{n \to \infty} \exp \left\{ \int_0^{1} \ln \left( 1 - \frac{X^2}{\pi^2 (t \cdot n)^2} \right) \, dt \right\} = 0
\]

(308)

1. The partition functions

\[
\lim_{n \to \infty} \left\{ \left( \int_0^{1} \mathcal{P}(t \cdot n) \cdot X^{t-n} \, dt \right)^{\frac{1}{n}} \right\} = \lim_{n \to \infty} \left\{ \exp \left\{ \int_0^{1} \ln \left( 1 - X^{t-n} \right) \right\} \right\}
\]

(309)

\[
= \lim_{n \to \infty} e^{\frac{s^2 - \Theta_2(X^n)}{\bar{h}(\log(Xn))}}
\]

(310)

so that

\[
\lim_{n \to \infty} \left\{ \int_0^{1} \mathcal{P}(t \cdot n) \cdot X^{t-n} \, dt \right\} = \lim_{n \to \infty} \left\{ \frac{1}{n} e^{\frac{s^2 - \Theta_2(X^n)}{\bar{h}(\log(Xn))}} \right\}
\]

(311)

for \( |X| < 1 \) we obtain that

\[
\lim_{n \to \infty} \left\{ \int_0^{1} \mathcal{P}(t \cdot n) \cdot X^{t-n} \, dt \right\} = e^{\frac{s^2}{\bar{h}(\log(X))^{-1}}} = \lim_{n \to \infty} \left\{ \frac{1}{n} \left( 1 - \left( \frac{\pi^2}{6} \log(X^n) \right)^n \right) \right\}
\]

(312)
\[
\lim_{n \to \infty} \left\{ \int_0^1 P(t \cdot n) \cdot X^{t \cdot n} dt \right\} = \int_0^1 \left( \frac{\zeta(2)}{\log (X^{-n})} \right)^{t \cdot n} dt \quad (313)
\]

So it seems as if
\[
P(t \cdot n) \cdot X^{t \cdot n} \approx \left( \frac{\zeta(2)}{\log (X^{-n})} \right)^{t \cdot n} \quad (314)
\]

2. Euler pentagonal Identity/and more general similar expression
\[
\prod_{1 \leq k \leq \infty} (1 - e^{-i \cdot s \cdot k}) = 2 \sum_{k \in \mathbb{N}} \cos \left( s \frac{3k^2 - k}{2} + k \cdot \pi \right) \quad (315)
\]
\[
\Rightarrow \lim_{n \to \infty} \exp \left\{ \int_0^1 \ln (1 - e^{-i \cdot s \cdot t \cdot n}) dt \right\} = \lim_{n \to \infty} \left\{ \int_0^1 \cos \left( \frac{s}{2} \left( 3 (t \cdot n)^2 - t \cdot n \right) + t \cdot \pi \right) \right\} \quad (316)
\]

3. Jacobi’s Identity

4. Cauchy result in Andrew’s book

4 \ A-Hypergeometrics for integer factoring

Consider the binary expansion of a prime number
\[
p(2) = \sum_{0 \leq k \leq \lfloor \log_2(p(2)) \rfloor} a_k 2^k \quad (317)
\]
where \(a_k \in \{0, 1\}\). The fact that \(p\) is prime implies that \(p\) can not be factored into products of binary expansion i.e. the polynomial
\[
p(x) = \sum_{0 \leq k \leq \lfloor \log_2(p(2)) \rfloor} a_k x^k \quad (318)
\]
is irreducible over the field \(\{0, 1\}\) with coefficients all greater or equal to 0. Let us define an equivalence class for binary polynomials with positive coefficients.
\[
p(x) = \sum_{0 \leq k \leq \lfloor \log_2(p(2)) \rfloor} a_k x^k; \ a_k \in \{0, 1\} \quad (319)
\]
and
\[
q(x) = \sum_{0 \leq k \leq m} b_k x^k; \ 0 \leq b_k \leq 2^{\lfloor \log_2(p(2)) \rfloor} \quad (320)
\]
are polynomial with polynomial coefficients are in the same equivalent class if \(p(x)\) can be turned into \(q(x)\) by a sequence of changes of the form
\[
\forall 1 \leq k \leq n \quad x^k \leftrightarrow 2 \cdot x^{k-1} \quad (321)
\]
The equivalence class can be thought of as the set of polynomials with positive integer coefficients that evaluates to \( p(2) \). Furthermore, the polynomials are in one-to-one correspondence with integer points on the surface

\[
p(2) = \left( \sum_{0 \leq k \leq \lfloor \log_2(p(2)) \rfloor} a_k 2^k \right)
\]

where the \( a_k \)'s would correspond to coordinates of the points. The degree of \( q(x) \) is bounded above by \( \lfloor \log_2(p(2)) \rfloor \), the size of the equivalence class is \( O(\log_2(p(2))!) \). Furthermore, the primality of \( p(2) \) implies that all the polynomials in the equivalence class of \( p(x) \) are irreducible over \( \mathbb{Z}[x] \). This observation induces a factorization algorithm which does not involve the integer division operation.

**Theorem**

A number \( n(2) \) is a composite number if and only if there is exactly one element of the equivalence class of \( n(x) \) which is reducible over \( \mathbb{Z}[x] \).

The proof of this theorem follows from the fundamental theorem of algebra. In that if there were no member of the equivalence class of \( n(x) \) which were reducible, we would conclude that \( n(2) \) is prime if there were more than 2 distinct elements which were reducible then \( n(2) \) would have distinct prime factorizations.

Unfortunately, the size of the equivalence classes makes the run time of this algorithm somewhat worse than the usual factorization of integers by division. We propose to improve the running time of the algorithm by some how avoiding to explore the whole equivalence class but a just a small proportion of it. For this we express integer factorization as an optimization problem.

Our starting point is of course the polynomial induced by the binary expansion of the integer \( n \) given by

\[
n(x) = \sum_{0 \leq k \leq \lfloor \log_2(n(2)) \rfloor} n_k \cdot x^k; \ n_k \in \{0, 1\}
\]

By [Strumfels] we know that the \( \lfloor \log_2(n(2)) \rfloor \) (not necessarily distinct) roots of the equation

\[
n(x) = 0
\]

denoted

\[
x_k \left( n_0, n_1, \ldots, n_k, \ldots, n_{\lfloor \log_2(n(2)) \rfloor - 1} \right), \ \forall 1 \leq k \leq \lfloor \log_2(n(2)) \rfloor
\]

are hypergeometric functions of the polynomial’s coefficients \( \{n_k\}_{0 \leq k \leq \lfloor \log_2(n(2)) \rfloor} \).

So we want to do is to consider elementary symmetric polynomials \( S_1 (\alpha_1 \cdot x_1, \alpha_2 \cdot x_2, \ldots, \alpha_{\lfloor \log_2(n(2)) \rfloor} \cdot x_{\lfloor \log_2(n(2)) \rfloor} \).
in the roots where $\alpha \in \{0, 1\}^{\lfloor \log_2(n(2)) \rfloor}$, $\alpha \neq (0, 0, \cdots, 0)$ and we want to solve
the following Optimization problem

$$\min_{\alpha \in \{0, 1\}^{\lfloor \log_2(n(2)) \rfloor}} \{||\alpha||_{\ell_0}\}$$

subject to the constraints

$$S_t \left( \alpha_1 \cdot x_1, \alpha_2 \cdot x_2, \cdots, \alpha_{\lfloor \log_2(n(2)) \rfloor} \cdot x_{\lfloor \log_2(n(2)) \rfloor} \right) \in \{0, 1\}$$

(326)

The exploration of the equivalence classes is expressed by the following valid transforms

$$\forall 1 \leq k \leq \lfloor \log_2(n(2)) \rfloor \quad x^k \leftrightarrow 2 \cdot x^{k-1}$$

(327)

that is for $a_k > 0$.

$$x_k \left( a_0, a_1, \cdots, a_{k-1}, a_k, \cdots, a_{\lfloor \log_2(n(2)) \rfloor-1} \right) \rightarrow x_k \left( a_0, a_1, \cdots, 2a_k + a_{k-1}, 0, \cdots, a_{\lfloor \log_2(n(2)) \rfloor-1} \right)$$

(328)

Using the A.K.S. algorithm we can guarantee the existence of a unique solution.

Let $C[q(x)]$ denote the companion matrix of $q(x)$ a polynomials in the equivalence class of $n(x)$, let $V \left( x_1, x_2, \cdots, x_{\lfloor \log_2(n(2)) \rfloor} \right)$ denotes the vandermone matrix associated with the roots of $q(x)$ we have :

$$Tr \left\{ \left( C[q_{\alpha}(x)] \right)^k \right\} = \left( \sum_{1 \leq j \leq \lfloor \log_2(n(2)) \rfloor} \alpha_j \cdot \left[ a_0, a_1, \cdots, a_{k-1}, a_k, \cdots, a_{\lfloor \log_2(n(2)) \rfloor-1} \right] \right)^k$$

(329)

let $t_{\alpha} = \left( Tr \left\{ \left( C[q_{\alpha}(x)] \right)^k \right\} \right)_{1 \leq k \leq \lfloor \log_2(n(2)) \rfloor}$, were $q_{\alpha}(x)$ represents the polynomials with scaled root we have

$$\alpha = \left[ V \left( x_1, x_2, \cdots, x_{\lfloor \log_2(n(2)) \rfloor} \right) \right]^{-1} t_{\alpha}$$

(330)

the goal in this particular formulation is to estimate $t_{\alpha}$ and explores the space of polynomials in same equivalence class.
Example:

\[
\begin{align*}
2(2) &= 2 \\
3(2) &= 2 + 1 \\
4(2) &= 2^2 \\
5(2) &= 2^2 + 1 \\
6(2) &= 2^2 + 2 \\
7(2) &= 2^2 + 2 + 1 \\
8(2) &= 2^3 \\
9(2) &= 2^3 + 1 \\
10(2) &= 2^3 + 2 \\
11(2) &= 2^3 + 2 + 1 \\
12(2) &= 2^3 + 2^2 \\
13(2) &= 2^3 + 2^2 + 1 \\
14(2) &= 2^3 + 2^2 + 2 \\
15(2) &= 2^3 + 2^2 + 2 + 1 \\
16(2) &= 2^4 \\
17(2) &= 2^4 + 1 \\
18(2) &= 2^4 + 2 \\
19(2) &= 2^4 + 2 + 1 \\
20(2) &= 2^4 + 2^2 \\
21(2) &= 2^4 + 2^2 + 1 \\
22(2) &= 2^4 + 2^2 + 2 \\
23(2) &= 2^4 + 2^2 + 2 + 1 \\
24(2) &= 2^4 + 2^3 \\
25(2) &= 2^4 + 2^3 + 1 \\
26(2) &= 2^4 + 2^3 + 2 \\
27(2) &= 2^4 + 2^3 + 2 + 1
\end{align*}
\]

As we shall illustrate our scheme allows for reverse factoring of integers. We will illustrate the steps of the algorithm by factoring integers in the range \([9, 27]\).

Factoring 21

\[
21(2) \equiv 2^4 + 2^2 + 1 \Rightarrow 21(x) = x^4 + x^2 + 1.
\]

Let us first focus on the polynomials of degree 3 that are in the equivalence classe of 21(x). Their general expression will be of the form

\[
x^3a + x^2b + xc + d = (x \cdot \sqrt[3]{a})^3 + (x \cdot \sqrt[3]{a})^2 \cdot \frac{b}{\sqrt[3]{a}} + (x \cdot \sqrt[3]{a}) \cdot \frac{c}{\sqrt[3]{a}} + d
\]

Let us carry the following change of variable

\[
x = y - \left( \frac{b}{3 (\sqrt[3]{a})^2} \right)
\]

It then follows that

\[
x^3a + x^2b + xc + d = y^3 + py + q
\]
where
\[ p(a, b, c) = \frac{1}{3} \left( \frac{b}{\sqrt[3]{a}} \right)^2 + \frac{c}{\sqrt[3]{a}} \]  
(336)
\[ q(a, b, c, d) = \frac{2}{27} \left( \frac{b}{\sqrt[3]{a}} \right)^3 - \frac{1}{3} \left( \frac{b}{\sqrt[3]{a}} \right)^2 \left( \frac{c}{\sqrt[3]{a}} \right) + d \]  
(337)

The solution to the algebraic equation is expressed by
\[ x_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \]  
(338)
\[ x_2 = e^{i\frac{2\pi}{3}} \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + e^{i\frac{2\pi}{3}} \left( \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \right) \]  
(339)
\[ x_3 = e^{i\frac{2\pi}{3}} \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + e^{i\frac{2\pi}{3}} \left( \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \right) \]  
(340)

5 Notes on the Goldbach Conjecture

Goldbach conjecture and the partition function
\[ \sum_{n,m} \mathcal{P}'_{m,n} x^m y^n = \prod_{k \in \mathbb{N}^*} \left( 1 - x \cdot y^{p(k)} \right)^{-1} \]  
(341)
\[ = \prod_{k \in \mathbb{N}^*} \left( \sum_{l \in \mathbb{N}} x^l \cdot y^{p(k) \cdot l} \right) \]  
(342)
\[ \mathcal{P}'_{m,n} \text{ counts the number of partition of the number } n \text{ in } m \text{ prime parts}. \]  
(343)

Furthermore we may consider the following functional expression
\[ \prod_{k \in \mathbb{N}} \left[ \left( 1 - e^{s \cdot \ln(p(k))} \right)^{-1} - 1 + \left( 1 - e^{s \cdot \ln(p(-k))} \right)^{-1} \right] = \sum_{\alpha \in \mathbb{N}^*} e^{s \sum_{k \in \mathbb{N}} \alpha(p(k)) \cdot \ln(p(k))} \]  
(344)
\[ \prod_{k \in \mathbb{N}} \left[ \left( \sum_{n \in \mathbb{N}} e^{s \cdot (n \cdot \ln(p(k)))} \right) - 1 + \left( \sum_{m \in \mathbb{N}} e^{s \cdot (m \cdot \ln(p(-k)))} \right) \right] = \sum_{\beta \in \mathbb{Q}^*} e^{s \sum_{k \in \mathbb{N}} \ln(\beta_k)} \]  
(345)
\begin{align*}
\prod_{k \in \mathbb{N}} \left[ \left( \sum_{n>0} e^{s(n \cdot \ln(p(k)))} \right) + 1 + \left( \sum_{m>0} e^{s(m \cdot \ln(p(k)))} \right) \right] &= \sum_{\beta \in \mathbb{Q}} e^{s \cdot \ln(\beta_k)} \quad (346) \\
\prod_{k \in \mathbb{N}} \left[ \left( \sum_{n>0} p^{n \cdot s}(k) \right) + 1 + \left( \sum_{m>0} p^{m \cdot s}(-k) \right) \right] &= \sum_{\beta \in \mathbb{Q}} \beta^s \quad (347) \\
\prod_{k \in \mathbb{N}} \left[ \left( \sum_{n>0} p^{n \cdot s}(k) \right) + 1 + \left( \sum_{m>0} p^{m \cdot s}(-k) \right) \right] &= \left[ \left( \frac{d}{dx} \right)^s \sum_{\beta \in \mathbb{Q}} e^{\beta x} \right]_{x=0} \quad (348) \\
\lim_{n \to \infty} \exp \left\{ \int_0^1 \ln \left( \frac{2p^s(t \cdot n)}{1 - p^{2s}(t \cdot n)} \right) dt \right\} &= \left( \sum_{\beta \in \mathbb{Q}} \beta^s \right)^\frac{1}{n} \quad (349)
\end{align*}

6 Algorithmic aspects of the Experiments

Performing the run time analysis of the algorithm and possible improvement, discuss the platform

7 Analysis of the Experimental Results

How close case could we get to validating our conjectures/hypothesis, discussion of the implication of the experiements for the theoretical framework.

8 Conclusion

References