The Sieve of Eratosthenes and the Theorem of Goldbach

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§1. The theorem of Goldbach is well-known that one can write every even number as a sum of two prime numbers. In a letter of 1742, Euler has written: "I believe it is a completely acceptable theorem, although I cannot prove it." This theorem has still not been proved, and it is the same about the following theorem: The sequence of the twin prime numbers[1] is infinite. In an address delivered at the International Congress of Mathematics, Cambridge, 1912, E. Landau had said that he regarded these problems as "unattainable problems in modern science."

However, one has now a starting point for the treatment of these problems, after which one has discovered that the prime numbers of Goldbach and twin prime numbers can be determined by a method analogous to that of Eratosthenes. The first who had paid attention to this fact should be Jean Merlin.[2]

The method consists of a double employing the Eratosthenes sieve. Let us, for example, give the partition of the even number 26. We write the following two sequences of numbers

$$\underline{0}\quad 1\quad 2\quad 3\quad \underline{4}\quad 5\quad \underline{6}\quad 7\quad \underline{8}\quad 9\quad \underline{10}\quad 11\quad \underline{12}\quad 13\quad \underline{14}\quad \underline{15}\quad \underline{16}\quad 17\quad \underline{18}\quad 19\quad \underline{20}\quad \underline{21}\quad \underline{22}\quad 23\quad \underline{24}\quad \underline{25}\quad \underline{26}$$

$$\underline{26}\quad \underline{25}\quad \underline{24}\quad 23\quad \underline{22}\quad \underline{21}\quad \underline{20}\quad 19\quad \underline{18}\quad 17\quad \underline{16}\quad \underline{15}\quad \underline{14}\quad 13\quad \underline{12}\quad 11\quad \underline{10}\quad \underline{9}\quad \underline{8}\quad 7\quad \underline{6}\quad 5\quad \underline{4}\quad 3\quad 2\quad 1\quad \underline{0}\,.$$

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[1] That is to say that the couples of the prime numbers having the difference 2. See P. Stäckel in "Sitzungsberichte der Heidelberger Akademie Abt. A., Jahrg; 1916, 10 Abh.

[2] See Bulletin des Sciences mathematiques T. 39, I partie, 1915. See also Viggo Brun in "Archiv for Mathematik og Naturvidenskab" 1915, B. 34, nr. 8: "Über das Goldbachsche Gesetz und die Anzahl der Primzahlpaare."

The prime numbers not exceeding $\sqrt{26}$ are 2, 3 and 5. We efface the numbers of the form 2λ, 3λ and 5λ in our two sequences. The sum of a number of the first line and the number immediately below in the second line is 26. If these two numbers are not effaced, they are prime numbers, and give then a Goldbachian partition of 26. It is not necessary to write the second sequence. One can only choose the numbers 26 and 0 of the first sequence as the starting points of the effacements. By this method we obtain all the partitions of an even number x into a sum of two prime numbers lying between $\sqrt{x}$ and $x-\sqrt{x}$. On choosing 0 and 2 as the starting points, we can determine the twin prime numbers. We do not know if a treatment by this method can lead to a proof of these theorems; but we see that the method can at least lead to very profound results.

§2. We study at first the method of Eratosthenes, on giving it the following form:

Suppose that the series:

$$\begin{array}{cccccccccccl}
\underline{0} & \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} & \underline{7} & \underline{8} & \underline{9} & \underline{10} & \underline{\ldots x} \\
0 & & 2 & & 4 & & 6 & & 8 & & 10 & \ldots \\
0 & & & 3 & & & 6 & & & 9 & & \\
 & & \ldots & & & & & & & & & \\
0 & & & p_n & & & 2p_n & & & 3p_n & & \ldots \lambda p_n
\end{array}$$

are given, where x denotes an integer and p_n the n-th prime number:

$$p_n \leq \sqrt{x} < p_{n+1} \quad ,$$

and λ an integer:

$$\lambda p_n \leq x < (\lambda+1)p_n \quad .$$

The terms of the first series, which are different from all the terms of the other series, are the prime numbers lying between $\sqrt{x}$ and x and the number 1.

These are the terms not effaced by the Eratosthenes sieve. We generalize, on studying the following arithmetical progression

Δ	$\Delta + D$	$\Delta + 2D$	...
a_1	a_1+p_1	a_1+2p_1	...
	...		
a_r	a_r+p_r	a_r+2p_r	...

The progressions are extended from 0 to x. D denotes an integer prime to the prime numbers $p_1,\dots,p_r$ (successive or not, but different).

Δ and $a_1,\dots,a_r$ are integers:

$$0 < \Delta \le D \,, \qquad 0 < a_i < p_i \;.$$

We raise the following problem:

How many terms different from all the terms of the other lines does the first line contain?

We denote this number by

$$N(\Delta,\, D,\, x,\, a_1,\, p_1,\dots,a_r,\, p_r)$$

or often more briefly by

$$N(D,\, x,\, p_1,\dots,p_r) \;.$$

We obtain the fundamental formula:

$$\begin{aligned} &N(\Delta,\, D,\, x,\, a_1,\, p_1,\dots,a_r,\, p_r) \\ &= N(\Delta,\, D,\, x,\, a_1,\, p_1,\dots,a_{r-1},\, p_{r-1}) \\ &\quad - N(\Delta',\, Dp_r,\, x,\, a_1,\, p_1,\dots,a_{r-1},\, p_{r-1}) \;, \end{aligned}$$

where

$$0 < \Delta' \le Dp_r$$

or more briefly

$$N(D,\ x,\ p_1,\ldots,p_r) = N(D,\ x,\ p_1,\ldots,p_{r-1}) - N(Dp_r,\ x,\ p_1,\ldots,p_{r-1}) \qquad (1)$$

on studying at first our arithmetical progressions up to the progression $a_{r-1} + \lambda p_{r-1}$, and on subjoining then the progression $a_r + \lambda p_r$. Suppose that $N(\Delta,\ D,\ x,\ a_1,p_1,\ldots,a_{r-1},p_{r-1})$ is known. We deduce $N(\Delta,\ D,\ x,\ a_1,p_1,\ldots,a_r,p_r)$ from it on subtracting the number of the terms of the last progression, which are identical to the terms of the first progression, but not identical to the terms of the intermediate progressions.

We see that the number is equal to $N(\Delta',\ Dp_r,\ x,\ a_1,\ p_1,\ldots,p_{r-1})$ on noting that the terms of the last progression $a_r + \lambda p_r$, which are identical to the first progression $\Delta + \mu D$, are the terms between 0 and x of the arithmetical progression

$$\Delta' \qquad \Delta' + Dp_r \qquad \Delta' + 2Dp_r \ \ldots \ ,$$

where

$$0 < \Delta' \leq Dp_r \quad ,$$

Δ' being the smallest positive term of the progression.

The indeterminate equation

$$a_r + \lambda p_r = \Delta + \mu D$$

or

$$p_r\lambda - D\mu = \Delta - a_r$$

always has, as one knows, solutions, because p_r and D are relatively prime. The solutions are

$$\lambda = \lambda_0 + tD \ , \qquad \mu = \mu_0 + tp_r \quad ,$$

whenever λ_0, μ_0 are solutions and t runs through the values 0, $\pm 1, \pm 2, \ldots$.

The terms of the last progression, which are identical to the terms of the first progression, are then all the terms

$$a_r + \lambda p_r = a_r + \lambda_0 p_r + tDp_r \quad , \quad \text{where} \quad t = 0, \pm 1, \pm 2, \ldots \quad .$$

These are the terms of an arithmetical progression having the difference Dp_r.

We define particularly $N(\Delta, D, x)$ or briefly $N(D, x)$ as the numbers of the terms between 0 and x of the progression

$$\Delta \quad \Delta + D \quad \Delta + 2D \; . \; . \; . \; \Delta + \lambda D \quad ,$$

where

$$0 < \Delta \le D \; , \qquad \Delta + \lambda D \le x < \Delta + (\lambda+1)D \quad .$$

Hence we deduce that

$$\lambda + 1 = N(D, x) = \frac{x}{D} + \theta \; , \qquad \text{where} \quad -1 < \theta < 1 \quad .$$

We give an example, choosing

$\Delta = 2$ $D = 7$ $x = 60$ $a_1 = 2$ $p_1 = 2$ $a_2 = 1$ $p_2 = 3$ $a_3 = 4$ $p_3 = 5$

(A) 2 9 16 23 30 37 44 51 58

(B) 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52 54 56 58 60

(C) 1 4 7 10 13 16 19 22 25 28 31 34 37 40 43 46 49 52 55 58

(D) 4 9 14 19 24 29 34 39 44 49 54 59

The numbers of (A) which are different from the numbers of (B) and (C) are 9, 23, 51. We subjoin then the progression (D). The numbers of (A) and (D), which are identical, are 9 and 44, having the difference $7 \cdot 5$. We obtain then

$$N(7, 60, 2, 3, 5) = N(7, 60, 2, 3) - N(7 \cdot 5, 60, 2, 3)$$

or $2 = 3 - 1$.

From the formula (1) we deduce the following

$$N(D, x, p_1, \ldots, p_r) = N(D, x) - N(Dp_1, x) - N(Dp_2, x, p_1) - \ldots - N(Dp_r, x, p_1, \ldots, p_{r-1}) \tag{2}$$

and

$$N(D, x, p_1,\dots,p_r) = N(D, x) - N(Dp_1, x) - \dots - N(Dp_r, x)$$
$$+ N(Dp_2p_1, x)$$
$$+ N(Dp_3p_1, x) + N(Dp_3p_2, x, p_1)$$
$$+ \dots$$
$$+ N(Dp_rp_1, x) + N(Dp_rp_2, x, p_1)$$
$$+ \dots + N(Dp_rp_{r-1}, x, p_1,\dots,p_{r-2}) \tag{3}$$

We give the last formula a concise form

$$N(D, x, p_1,\dots,p_r) = N(D, x) - \sum_{a \le r} N(Dp_a, x)$$
$$+ \sum_{a \le r} \sum_{b < a} N(Dp_ap_b, x, p_1,\dots,p_{b-1}). \tag{3'}$$

When the question is to determine a lower bound for $N(D, x, p_1,\dots,p_r)$ we can set aside as many positive terms as we want in the formula (3). One can choose these terms in several different ways[3], for example, the terms which lie on the right of a vertical line. In general we obtain the formula

$$N(D, x, p_1,\dots,p_r) > N(D, x) - \sum_{a \le r} N(Dp_a, x)$$
$$+ \sum_{\omega_1}\sum N(Dp_ap_b, x, p_1,\dots,p_{b-1}) , \tag{4}$$

where we have chosen for p_ap_b a domain ω_1 which lies in the interior of the following domain

$$\begin{array}{llll} p_2p_1 & & & \\ p_3p_1 & p_3p_2 & & \\ \dots & & & \\ p_rp_1 & p_rp_2 & \cdots & p_rp_{r-1} \end{array}$$

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[3] See: "Nyt tidsskrift" 1918: Une formule exacte pour la determination du nombre des nombres premiers audessous de x, etc. by Viggo Brun.

On applying the formula (4) twice we obtain the new formula

$$N(D, x, p_1,\dots,p_r) > N(D, x) - \sum_{a \le r} N(Dp_a, x)$$

$$+ \sum_{\omega_1}\sum \left(N(Dp_a p_b, x) - \sum_{c<b} N(Dp_a p_b p_c, x) \right)$$

$$+ \sum_{\omega_1'}\sum \; \sum_{\omega_2}\sum N(Dp_a p_b p_c p_d, x, p_1,\dots,p_{d-1}) \; ,$$

where $\omega_1' \le \omega_1$ and ω_2 denotes the domain for $p_c p_d$.

On continuing and applying

$$N(d, x) = \frac{x}{d} + \theta, \qquad \text{where} \quad -1 < \theta < 1 \; ,$$

we obtain at last the general formula

$$\frac{D}{x} N(D, x, p_1,\dots,p_r) > 1 - \sum_{a \le r} \frac{1}{p_a} + \sum_{\omega_1}\sum \frac{1}{p_a p_b}\left(1 - \sum_{c<b} \frac{1}{p_c}\right)$$

$$+ \sum_{\omega_1'}\sum \; \sum_{\omega_2}\sum \frac{1}{p_a p_b p_c p_d}\left(1 - \sum_{e<d} \frac{1}{p_e}\right) + \dots - \frac{RD}{x} \quad , \tag{5}$$

where R denotes the number of terms, and where $\omega_1' \le \omega_1$ etc.

We can also give the formula (5) the following form, on supposing particularly $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ etc.:

$$N(D, x, 2, 3, 5,\dots,p_r) > \frac{x}{D}\Bigg[1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} - \dots - \frac{1}{p_r}$$

$$+ \frac{1}{3\cdot 2}$$

$$+ \frac{1}{5\cdot 2} + \frac{1}{5\cdot 3}\left(1 - \frac{1}{2}\right)$$

$$+ \frac{1}{7\cdot 2} + \frac{1}{7\cdot 3}\left(1 - \frac{1}{2}\right) + \frac{1}{7\cdot 5}\begin{pmatrix} 1 - \frac{1}{2} - \frac{1}{3} \\ + \frac{1}{3\cdot 2} \end{pmatrix}$$

$$+ \dots$$

$$+ \frac{1}{p_r \cdot 2} + \frac{1}{p_r \cdot 3}(1-\frac{1}{2}) + \frac{1}{p_r \cdot 5}\begin{pmatrix} 1-\frac{1}{2}-\frac{1}{3} \\ +\frac{1}{3 \cdot 2} \end{pmatrix}$$

$$+ \frac{1}{p_r \cdot 7}\left(\begin{array}{l} 1-\frac{1}{2}-\frac{1}{3}-\frac{1}{5} \\ +\frac{1}{3\cdot 2} \\ +\frac{1}{5\cdot 2}+\frac{1}{5\cdot 3}(1-\frac{1}{2}) \end{array} \right) + \ldots \Bigg] - R ,$$

where one can set aside every term (the subsequent parenthesis included), which follows the sign + .

R denotes the number of terms employed.

We obtain the better lower bound for N, when we aside those terms, which multiplied by $\frac{x}{D}$ are less than the number of terms employed.

We give an example, choosing $x = 1{,}000$, $D = 1$ and $p_r = 31$ which is the greatest prime number not exceeding $\sqrt{x}$.

$$N(1,\, 10^3,\, 2,\, 3, \ldots, 31) > 10^3 \Big[1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{31} + \frac{1}{3\cdot 2}$$

$$+ \frac{1}{5\cdot 2} + \frac{1}{5\cdot 3}(1-\frac{1}{2}) + \frac{1}{7\cdot 2} + \frac{1}{7\cdot 3}(1-\frac{1}{2}) + \frac{1}{7\cdot 5}(1-\frac{1}{2}-\frac{1}{3}+\frac{1}{3\cdot 2})$$

$$+ \frac{1}{11\cdot 2} + \frac{1}{11\cdot 3}(1-\frac{1}{2}) + \frac{1}{11\cdot 5}(1-\frac{1}{2}-\frac{1}{3}+\frac{1}{3\cdot 2})$$

$$+ \frac{1}{13\cdot 2} + \frac{1}{13\cdot 3}(1-\frac{1}{2}) + \frac{1}{13\cdot 5}(1-\frac{1}{2}-\frac{1}{3}+\frac{1}{3\cdot 2})$$

$$+ \frac{1}{17\cdot 2} + \frac{1}{17\cdot 3}(1-\frac{1}{2}) + \frac{1}{19\cdot 2} + \frac{1}{19\cdot 3}(1-\frac{1}{2}) + \frac{1}{23\cdot 2} + \frac{1}{23\cdot 3}(1-\frac{1}{2})$$

$$+ \frac{1}{29\cdot 2} + \frac{1}{29\cdot 3}(1-\frac{1}{2}) + \frac{1}{31\cdot 2} + \frac{1}{31\cdot 3}(1-\frac{1}{2}) \Big] - 52$$

We have set aside the term $\frac{1}{17\cdot 5}(1-\frac{1}{2}-\frac{1}{3}-\frac{1}{3\cdot 2}) = 0.0039\ldots$

since $10^3 \cdot 0.0039\ldots = 3.9\ldots$ is less than 4, the number of terms employed. In the term $\frac{1}{11\cdot 7}\left(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{3\cdot 2} + \frac{1}{5\cdot 2} + \frac{1}{5\cdot 3}\,(1 - \frac{1}{2})\right)$, we would at first set aside $\frac{1}{5\cdot 3}\,(1 - \frac{1}{2})$ since $\frac{10^3}{11\cdot 7\cdot 5\cdot 3}\,(1 - \frac{1}{2}) = 0.4\ldots$ is less than 2, and we should also set aside the term $\frac{1}{11\cdot 7}\left(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{3\cdot 2} + \frac{1}{5\cdot 2}\right) = 0.003\ldots$ since $10^3 \cdot 0.003\ldots = 3.\ \ldots$ is less than 6.

We obtain then

$$N(1,\ 10^3,\ 2,\ 3,\ldots,31) > 109 - 52 = 57 \quad .$$

We can express this result in the following way:

When we efface among 1,000 numbers all the multiples of two, three, five up to 31, there remain still at least 57 numbers. Thence we deduce particularly that there exist more than 56 prime numbers between 31 and 1000, on observing that

$$N(1,\ 10^3,\ 2,\ 3,\ldots,31) = \pi(10^3) - \pi(\sqrt{10^3}) + 1$$

when we choose 0 as the starting point of the effacements.

Here $\pi(x)$ denotes the number of prime numbers not exceeding x.

Here we have chosen the domains ω in a way to obtain the most suitable lower bound. If we choose the domains ω by the same principle, we find

$N(1,\ 10^3,\ 2,\ 3,\ldots,31) > 109 - 52 = 57,$

$\qquad$ while $\pi(10^3) - \pi(\sqrt{10^3}) = 158$,

$N(1,\ 10^4,\ 2,\ 3,\ldots,97) > 820 - 284 = 536$,

$\qquad$ while $\pi(10^4) - \pi(\sqrt{10^4}) = 1{,}206$,

$N(1,\ 10^5,\ 2,\ 3,\ldots,313) > 5{,}733 - 1{,}862 = 3{,}871$,

$\qquad$ while $\pi(10^5) - \pi(\sqrt{10^5}) = 9{,}528$.

In the sequel we will choose the domains ω by simpler principles.

To illustrate the principles sought after we give at first three examples:

Eg. 1) $$N(1, x, 2, 3, 5, 7) > x\left[1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{3\cdot 2} + \frac{1}{5\cdot 2} + \frac{1}{5\cdot 3}\left(1 - \frac{1}{2}\right) + \frac{1}{7\cdot 2} + \frac{1}{7\cdot 3}\left(1 - \frac{1}{2}\right) + \frac{1}{7\cdot 5}\begin{pmatrix} 1 - \frac{1}{2} - \frac{1}{3} \\ \\ + \frac{1}{3\cdot 2} \end{pmatrix}\right] - 16$$

$$= x\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right) - 2^4$$

We have set aside no terms.

Eg. 2) $$N(1, x, 2, 3, 5, 7, 11) > x\left[1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{3\cdot 2} + \frac{1}{5\cdot 2} + \frac{1}{5\cdot 3}\left(1 - \frac{1}{2}\right) + \frac{1}{7\cdot 2} + \frac{1}{7\cdot 3}\left(1 - \frac{1}{2}\right) + \frac{1}{7\cdot 5}\left(1 - \frac{1}{2} - \frac{1}{3}\right) + \frac{1}{11\cdot 2} + \frac{1}{11\cdot 3}\left(1 - \frac{1}{2}\right) + \frac{1}{11\cdot 5}\left(1 - \frac{1}{2} - \frac{1}{3}\right) + \frac{1}{11\cdot 7}\left(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5}\right)\right] - 26 \quad ,$$

where the terms set aside are added on a small scale. One can also write

$$N(1, x, 2, 3, 5, 7, 11) > x\left[\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right)\left(1 - \frac{1}{11}\right) - \left(\frac{1}{7\cdot 5\cdot 3\cdot 2} + \frac{1}{11\cdot 5\cdot 3\cdot 2} + \frac{1}{11\cdot 7\cdot 3\cdot 2} + \frac{1}{11\cdot 7\cdot 5\cdot 2} + \frac{1}{11\cdot 7\cdot 5\cdot 3}\right) + \left(\frac{1}{11\cdot 7\cdot 5\cdot 3\cdot 2}\right)\right] - \left(1 + 5 + \frac{5\cdot 4}{1\cdot 2} + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}\right)$$

$$= x\left[0.2078 - 0.0121 + 0.0004\right] - 26 = 0.1961x - 26.$$

Here we have set aside all terms of the form $\frac{1}{p_a p_b p_c p_d}$ and of the form $\frac{1}{p_a p_b p_c p_d p_e}$.

Eg. 3) $N(1, x, 2, 3, 5, 7, 11, 13, 17, 19) > x\Big[1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5}$

$$- \frac{1}{7} - \frac{1}{11} - \frac{1}{13} - \frac{1}{17} - \frac{1}{19} + \frac{1}{3 \cdot 2} + \frac{1}{5 \cdot 2} + \frac{1}{5 \cdot 3}\left(1 - \frac{1}{2}\right) + \frac{1}{7 \cdot 2}$$

$$+ \frac{1}{7 \cdot 3}\left(1 - \frac{1}{2}\right) + \frac{1}{7 \cdot 5}\left(1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{3 \cdot 2}\right)$$

$$+ \frac{1}{11 \cdot 2} + \frac{1}{11 \cdot 3}\left(1 - \frac{1}{2}\right) + \frac{1}{11 \cdot 5}\begin{pmatrix} 1 - \frac{1}{2} - \frac{1}{3} \\ + \frac{1}{3 \cdot 2} \end{pmatrix} + \frac{1}{11 \cdot 7}\begin{pmatrix} 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} \\ + \frac{1}{3 \cdot 2} \\ + \frac{1}{5 \cdot 2} \end{pmatrix}$$

$$+ \frac{1}{13 \cdot 2} + \frac{1}{13 \cdot 3}\left(1 - \frac{1}{2}\right) + \frac{1}{13 \cdot 5}\begin{pmatrix} 1 - \frac{1}{2} - \frac{1}{3} \\ + \frac{1}{3 \cdot 2} \end{pmatrix} + \frac{1}{13 \cdot 7}\begin{pmatrix} 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} \\ + \frac{1}{3 \cdot 2} \\ + \frac{1}{5 \cdot 2} \end{pmatrix}$$

$$+ \frac{1}{17 \cdot 2} + \frac{1}{17 \cdot 3}\left(1 - \frac{1}{2}\right) + \frac{1}{17 \cdot 5}\begin{pmatrix} 1 - \frac{1}{2} - \frac{1}{3} \\ + \frac{1}{3 \cdot 2} \end{pmatrix} + \frac{1}{17 \cdot 7}\begin{pmatrix} 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} \\ + \frac{1}{3 \cdot 2} \\ + \frac{1}{5 \cdot 2} \end{pmatrix}$$

$$+ \frac{1}{19 \cdot 2} + \frac{1}{19 \cdot 3}\left(1 - \frac{1}{2}\right) + \frac{1}{19 \cdot 5}\begin{pmatrix} 1 - \frac{1}{2} - \frac{1}{3} \\ + \frac{1}{3 \cdot 2} \end{pmatrix} + \frac{1}{19 \cdot 7}\begin{pmatrix} 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} \\ + \frac{1}{3 \cdot 2} \\ + \frac{1}{5 \cdot 2} \end{pmatrix}\Big] - 72$$

$$= 0.163x - 72 .$$

Here we have set aside the terms on the right of the vertical lines. One see that the expression is of the form

$$1 - \sum \frac{1}{p_a} + \sum\sum \frac{1}{p_a p_b} - \sum\sum\sum \frac{1}{p_a p_b p_c} + \sum\sum\sum\sum \frac{1}{p_a p_b p_c p_d} ,$$

where p_a, p_b, p_c and p_d run through the following values

p_a	2	3	5	7	11	13	17	19
p_b	2	3	5	7				
p_c	2	3	5	7				
p_d	2							

in which $a > b > c > d$.

§3. We study at first the method employed for example 2.

We do not apply the general formula (5), but we deduce directly from the formula (3'):

$$N(D, x, p_1, \ldots, p_r) = N(D, x) - \sum_{a \leq r} N(Dp_a, x)$$
$$+ \sum_{a \leq r} \sum_{b < a} N(Dp_a p_b, x, p_1, \ldots, p_{b-1}) .$$

On employing this formula twice, we obtain

$$\begin{aligned} N(D, x, p_1, \ldots, p_r) = N(D, x) &- \sum_{a \leq r} N(Dp_a, x) \\ &+ \sum_{a \leq r} \sum_{b < a} N(Dp_a p_b, x) \\ &- \sum_{a \leq r} \sum_{b < a} \sum_{c < b} N(Dp_a p_b p_c, x) \\ &+ \sum_{a \leq r} \sum_{b < a} \sum_{c < b} \sum_{d < c} \\ &\cdot N(Dp_a p_b p_c p_d, x, p_1, \ldots, p_{d-1}) . \quad (6) \end{aligned}$$

The last sum is positive (or 0). On applying

$$N(d, x) = \frac{x}{d} + \theta , \qquad \text{where } -1 \leq \theta < 1 ,$$

thence we conclude:

$$N(D,\ x,\ p_1,\dots,p_r) > \frac{x}{D}\left[1 - \sum_{a \le r} \frac{1}{p_a} + \sum_{a \le r}\ \sum_{b < a} \frac{1}{p_a p_b} - \sum_{a \le r}\ \sum_{b < a}\ \sum_{c < b} \frac{1}{p_a p_b p_c}\right] - R\ , \tag{7}$$

or more briefly

$$N(D,\ x,\ p_1,\dots,p_r) > \frac{x}{D}\left[1 - \Sigma_1 + \Sigma_2 - \Sigma_3\right] - R\ , \tag{7'}$$

where Σ_1 is equal to the sum of the terms of the first of the following three lines

$$\begin{array}{l} \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_r} = \sigma \\ \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_r} \\ \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_r}\ , \end{array} \tag{A}$$

Σ_2 is equal to the sum of the terms formed by multiplication of every term of the first line by those terms of the second line, which lie on the left of this term, and Σ_3 can be defined similarly.

We will say, in the sequel, that we calculate the expression

$$1 - \Sigma_1 + \Sigma_2 - \Sigma_3$$

by means of diagram (A) or more briefly by means of the diagram

r terms

three lines

We compare Σ_2 and σ^2:

$$\sigma^2 = \left(\frac{1}{p_1}\right)^2 + \left(\frac{1}{p_2}\right)^2 + \dots + \left(\frac{1}{p_r}\right)^2 + 2\Sigma_2 > 2\Sigma_2$$

$$\text{or} \quad \sigma\Sigma_1 > 2\Sigma_2\ .$$

We will also prove that

$$\sigma\textstyle\sum_2 > 3\textstyle\sum_3 \quad \text{or} \quad \left(\displaystyle\sum_{c \le r} \frac{1}{p_c}\right)\left(\displaystyle\sum_{a \le r}\ \sum_{b < a} \frac{1}{p_a p_b}\right)$$

$$> 3\left(\sum_{a \le r}\ \sum_{b < a}\ \sum_{c < b} \frac{1}{p_a p_b p_c}\right)$$

Any term $\dfrac{1}{p_\alpha p_\beta p_\gamma}$, where $\gamma < \beta < \alpha \le r$, is represented once in $\sum_3$ but, as we see, three times in $\sigma\sum_2$.

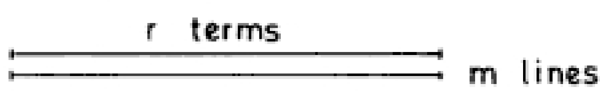
We search at first $\dfrac{1}{p_\alpha}$ in $\displaystyle\sum_{c \le r} \frac{1}{p_c}$ and $\dfrac{1}{p_\beta p_\gamma}$ in $\displaystyle\sum_{a \le r}\ \sum_{b < a} \frac{1}{p_a p_b}$, and then $\dfrac{1}{p_\beta}$ in $\displaystyle\sum_{c \le r} \frac{1}{p_c}$ and $\dfrac{1}{p_\alpha p_\gamma}$ in $\displaystyle\sum_{a \le r}\ \sum_{b < a} \frac{1}{p_a p_b}$, and at last $\dfrac{1}{p_\gamma}$ in $\displaystyle\sum_{c \le r} \frac{1}{p_c}$ and $\dfrac{1}{p_\alpha p_\beta}$ in $\displaystyle\sum_{a \le r}\ \sum_{b < a} \frac{1}{p_a p_b}$.

The term $\dfrac{1}{p_\alpha p_\beta p_\gamma}$ is therefore represented three times in $\sigma\sum_2$, which contains also terms of the form $\dfrac{1}{p_\alpha^2 p_\beta}$ etc. Hence we conclude that $\sigma\sum_2 > 3\sum_3$.

We can generalize the formula (7), on calculating the last sum in (6) by means (6). On continuing we obtain a formula analogous to (7) or more briefly analogous to (7'):

$$N(D,\ x,\ p_1,\dots,p_r) > \frac{x}{D}\left[1 - \textstyle\sum_1 + \sum_2 - \dots - \sum_m\right] - R\,, \qquad (8)$$

where m is an odd number satisfying $m \le r$, and where the expression $1 - \sum_1 + \sum_2 - \dots - \sum_m$ is calculated by means of the diagram



We can, in the special case $m = r$, calculate this expression:

$$1 - \sum_1 + \sum_2 - \ldots + (-1)^r \sum_r$$

$$= (1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_r})$$

$$= 1 - \sum_{a \leq r} \frac{1}{p_a} + \sum_{a \leq r} \sum_{b < a} \frac{1}{p_a p_b} - \ldots ,$$

where r may be even or odd. The number of terms is 2^r in this case. We obtain then the formula

$$N(D, x, p_1, \ldots, p_r) > \frac{x}{D} (1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_r}) - 2^r . \quad (9)$$

In general case we will determine a lower bound for the expression

$$1 - \sum_1 + \sum_2 - \ldots - \sum_m .$$

We can, as before, prove that

$$\sigma = \sum_1 , \quad \sigma \sum_i > (i+1) \sum_{i+1} \quad (1 \leq i \leq m-1)$$

whence $\sigma^m > m! \sum_m$.

Hence we conclude

$$\sum_m < \frac{\sigma}{m} \sum_{m-1} \quad (10)$$

and

$$\sum_m < \frac{\sigma^m}{m!} < \left(\frac{e\sigma}{m}\right)^m \quad (11)$$

on applying the Stirling formula

$$m! = \left(\frac{m}{e}\right)^m (\sqrt{2\pi m} + \theta) , \quad -1 < \theta < 1 .$$

We now write the formula (8) in a different way

$$N(D, x, p_1, \ldots, p_r) > \frac{x}{D} \Big[(1 - \sum_1 + \sum_2 - \ldots + (-1)^r \sum_r)$$

$$- (\sum_{m+1} - \sum_{m+2} + \ldots + (-1)^r \sum_r) \Big] - R .$$

We know the value of the first parenthesis in the form of a product. The second parenthesis is composed of a series of decreasing terms, whenever $m+2 > \sigma$, and then it has a value less than $\sum_{m+1}$, which is less than $\left(\frac{e\sigma}{m+1}\right)^{m+1}$.

We can therefore write

$$N(D, x, p_1,\dots,p_r) > \frac{x}{D}\left[(1-\frac{1}{p_1}) \dots (1-\frac{1}{p_r}) - (\frac{e\sigma}{m+1})^{m+1}\right] - R .$$

It is not difficult to determine the value of R[4]:

$$R = 1 + \binom{r}{1} + \binom{r}{2} + \dots + \binom{r}{m}$$

$$< 1 + r + r^2 + \dots + r^m < r^{m+1} .$$

We obtain then the formula

$$N(D, x, p_1,\dots,p_r) > \frac{x}{D}\left[(1-\frac{1}{p_1}) \dots (1-\frac{1}{p_r}) - (\frac{e\sigma}{m+1})^{m+1}\right] - r^{m+1} \tag{12}$$

whenever

$$m+2 > \sigma = \frac{1}{p_1} + \dots + \frac{1}{p_r} .$$

This formula is more useful than (9), the growth of r^{m+1} being not so great as that of 2^r. But the growth of the term R is still too great for our purpose.

§4. For this reason we shall choose the domains ω in another way, setting aside all terms on the right of the vertical lines, as in the example 3 (§2).

At first we set aside in the formula (3) all positive terms on the right on a vertical line. We obtain then the following formula

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[4] See, for example, Landau: Handbuch der Lehre von der Verteilung der Primzahlen, I, p. 67.

$$N(D, x, p_1,\dots,p_r) > N(D, x) - \sum_{a \le r} N(Dp_a, x) + \sum_{a \le r} \sum_{\substack{b < a \\ b < t}} N(Dp_a p_b, x, p_1,\dots,p_{b-1}) \,, \tag{13}$$

where t is an integer less than r.

The terms of the last sum can be calculated by means of the same formula, whence one deduces

$$N(D, x, p_1,\dots,p_r) > N(D, x) - \sum_{a \le r} N(Dp_a, x)$$

$$+ \sum_{a \le r} \sum_{\substack{b < a \\ b < t}} N(Dp_a p_b, x) - \sum_{a \le r} \sum_{\substack{b < a \\ b < t}} \sum_{\substack{c < b \\ c < t}} N(Dp_a p_b p_c, x)$$

$$+ \sum_{a \le r} \sum_{\substack{b < a \\ b < t}} \sum_{\substack{c < b \\ c < t}} \sum_{\substack{d < c \\ d < u}} N(Dp_a p_b p_c p_d, x, p_1,\dots,p_{d-1}) \,,$$

where u is an integer less than t.

On continuing, and on applying

$$N(d, x) = \frac{x}{d} + \theta \,, \qquad -1 < \theta < 1 \,,$$

we obtain at last the formula

$$N(D, x, p_1,\dots,p_r) > \frac{x}{D} \Bigg[1 - \sum_{a \le r} \frac{1}{p_r} + \sum_{a \le r} \sum_{\substack{b < a \\ b < t}} \frac{1}{p_a p_b}$$

$$- \sum_{a \le r} \sum_{\substack{b < a \\ b < t}} \sum_{\substack{c < b \\ c < t}} \frac{1}{p_a p_b p_c}$$

$$+ \sum_{a \le r} \sum_{\substack{b < a \\ b < t}} \sum_{\substack{c < b \\ c < t}} \sum_{\substack{d < c \\ d < u}} \frac{1}{p_a p_b p_c p_d} - \dots \Bigg] - R \tag{14}$$

or more briefly

$$N(D, x, p_1,\dots,p_r) > \frac{x}{D} \Big[1 - S_1 + S_2 - \dots - S_{2n-1} \Big] - R \,, \tag{14'}$$

where the expression

$$E_n = 1 - S_1 + S_2 - \dots - S_{2n-1}$$

is calculated by means of the diagram in the form of stairs

$$\begin{array}{l}
\overbrace{\frac{1}{p_1} + \dots + \frac{1}{p_{w-1}}}^{\sigma_n} + \dots + \overbrace{\frac{1}{p_u} + \dots + \frac{1}{p_{t-1}}}^{\sigma_2} \quad \overbrace{\frac{1}{p_t} + \dots + \frac{1}{p_r}}^{\sigma_1} \\
\frac{1}{p_1} + \dots + \frac{1}{p_{w-1}} + \dots + \frac{1}{p_u} + \dots + \frac{1}{p_{t-1}} \\
\frac{1}{p_1} + \dots + \frac{1}{p_{w-1}} + \dots + \frac{1}{p_u} + \dots + \frac{1}{p_{t-1}} \\
\dots\dots \\
\frac{1}{p_1} + \dots + \frac{1}{p_{w-1}} \\
\frac{1}{p_1} + \dots + \frac{1}{p_{w-1}} .
\end{array}$$

We choose the prime numbers of the diagram as successive prime numbers lying in the interior of the following intervals

$$R_r^{\frac{1}{\alpha^n}} \qquad p_1 \qquad p_r^{\frac{1}{\alpha^{n-1}}} \quad \dots \quad p_r^{\frac{1}{\alpha^2}} \qquad p_r^{\frac{1}{\alpha}} \qquad p_r$$

where $\alpha > 1$.

We apply the Mertens' formulas, giving them the following forms:

$$\sum_2^x \frac{1}{p} = \log\log x + 0.261\dots + \theta \frac{5}{\log x} , \qquad -1 < \theta < 1$$

$$\prod_2^x \left(1 - \frac{1}{p}\right) = e^{7\theta/\log x} \; \frac{0.561\dots}{\log x} , \qquad -1 < \theta < 1$$

where log denotes the natural logarithm.

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5) See, "Journal fur die reine und angewandte Mathematik" B.78, 1874, or Landau, Handbuch, I, p. 201.

Hence we conclude

$$\sum_{x}^{x^{\alpha}} \frac{1}{p} = \log \alpha + \theta \, \frac{5(1+\frac{1}{\alpha})}{\log x} \quad , \qquad \prod_{x}^{x^{\alpha}} \left(1-\frac{1}{p}\right) = \frac{1}{\alpha}\, e^{(1+1/\alpha)7\theta/\log x} .$$

But in that case we can choose p_1 sufficiently large for which

$$\sigma_1 = \frac{1}{p_t} + \ldots + \frac{1}{p_r} < \log \alpha_0 ,$$

$$\sigma_2 = \frac{1}{p_u} + \ldots + \frac{1}{p_{t-1}} < \log \alpha_0 , \ldots$$

$$\sigma_n = \frac{1}{p_1} + \ldots + \frac{1}{p_{w-1}} < \log \alpha_0 \tag{15}$$

and

$$\pi_1 = \left(1-\frac{1}{p_t}\right) \ldots \left(1-\frac{1}{p_r}\right) > \frac{1}{\alpha_0} ,$$

$$\pi_2 = \left(1-\frac{1}{p_u}\right) \ldots \left(1-\frac{1}{p_{t-1}}\right) > \frac{1}{\alpha_0} , \ldots$$

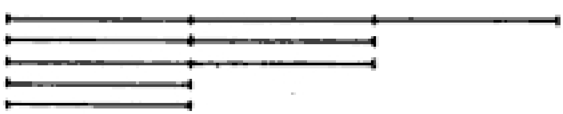
$$\pi_n = \left(1-\frac{1}{p_1}\right) \ldots \left(1-\frac{1}{p_{w-1}}\right) > \frac{1}{\alpha_0} , \tag{16}$$

whenever $\alpha_0 > \alpha$.

We suppose particularly $\log \alpha_0 < 1$.

We try to realize a successive calculation of the sums, to which the diagrams in the form of stairs give rise.

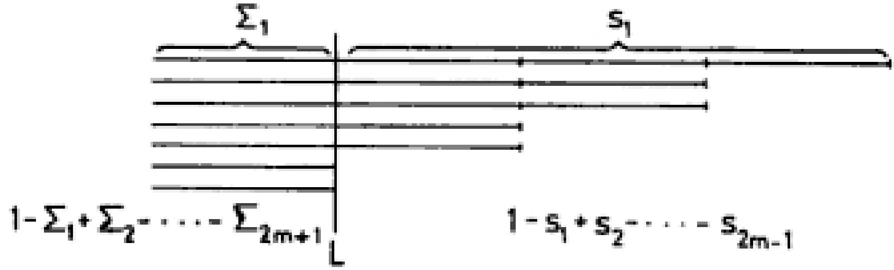
Suppose that we have calculated by means of the diagram



(2m-1) lines

giving rise to the expression $E_m = 1 - S_1 + S_2 - \ldots - S_{2m-1}$.

We subjoin then $2m+1$ lines on the left, (which only taken gives rise to the expression $1 - \sum_1 + \sum_2 - \ldots - \sum_{2m+1}$):



The sum $\sum \frac{1}{p_a}$ is now equal to $\sum_1 + S_1$. We see also that the new sum $\sum\sum \frac{1}{p_a p_b}$ is equal to $\sum_2 + S_1\sum_1 + S_2$ on studying the three possible cases:

p_a occurs on the left of L and p_b on the left of L $(\sum_2)$

p_a occurs on the left of L and p_b on the right of L $(S_1\sum_1)$

p_a occurs on the right of L and p_b on the right of L (S_2) .

In general we can calculate the new expression E_{m+1} by the following way:

$$E_{m+1} = 1 - \left(\sum_1 + S_1\right) + \left(\sum_2 + S_1\sum_1 + S_2\right) - \left(\sum_3 + S_1\sum_2 + S_2\sum_1 + S_3\right)$$

$$+ \ldots - \left(\sum_{2m+1} + S_1 \sum_{2m} + \ldots + S_{2m-1} \sum_2\right).$$

We compare this expression with the following product

$$(1 - \sum_1 + \sum_2 - \ldots \pm \sum_\nu)\ (1 - S_1 + S_2 - \ldots - S_{2m-1})$$

$$= 1 - (\sum_1 + S_1) + (\sum_2 + S_1\sum_1 + S_2) - \ldots$$

$$- (\sum_{2m+1} + S_1\sum_{2m} + \ldots + S_{2m-1} \sum_2)$$

$$+ (\sum_{2m+2} + S_1 \sum_{2m+1} + \ldots + S_{2m-1} \sum_3) - \ldots \quad .$$

The first factor contains as many terms as possible, that is to say, ν is equal to the number of the terms in $\sum_1$. The product contains, as one sees, all the terms of E_{m+1} and in addition a series of parentheses, whose values, by (10), are decreasing, since

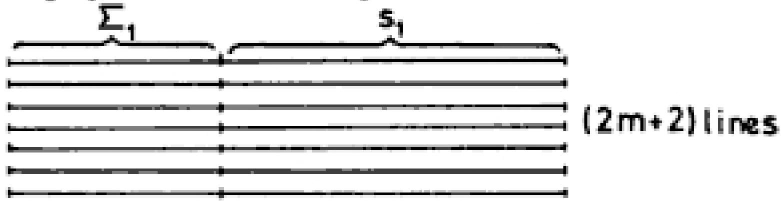
$\sum_1 = \sigma_{m+1} < \log \alpha_0 < 1$, and having alternative signs. Hence we conclude

$$E_{m+1} > \pi_{m+1} E_m - (E_{2m+2} + S_1 \textstyle\sum_{2m+1} + \dots + S_{2m-1} \textstyle\sum_3) . \qquad (17)$$

We can determine an upper bound for the last parenthesis. It is a sum of the different products of $(2m+2)$ numbers $\frac{1}{p}$, which all occur in the two sums S_1 and $\sum_1$. But we obtain the sum of all possible products of that form, on forming the sum

$$(S_1 + \textstyle\sum_1)_{2m+2}$$

calculating by means of the diagram



But by (11) and (15) we obtain

$$(S_1 + \textstyle\sum_1)_{2m+2} < \left(\frac{e(S_1 + \sum_1)}{2m+2}\right)^{2m+2} < \left(\frac{e(m+1)\log \alpha_0}{2(m+1)}\right)^{2m+2}$$

$$= \left(\frac{e \log \alpha_0}{2}\right)^{2m+2}$$

Our parenthesis (in (17)) is then still less, whence we conclude that

$$E_{m+1} > \pi_{m+1} E_m - \left(\frac{e \log \alpha_0}{2}\right)^{2m+2}$$

We obtain then particularly, since $E_1 = 1 - S_1$,

$$E_1 > 1 - \log \alpha_0 ,$$

$$E_2 > \pi_2 E_1 - \left(\frac{e \log \alpha_0}{2}\right)^4 > \pi_2 \left(1 - \log \alpha_0 - \alpha_0 \left(\frac{e \log \alpha_0}{2}\right)^4\right)$$

on applying (16). On continuing in the same way, we obtain at last

$$E_n > \pi_2 \pi_3 \cdots \pi_n \left(1 - \log \alpha_0 - \alpha_0\left(\frac{e \log \alpha_0}{2}\right)^4 - \cdots - \alpha_0^{n-1}\left(\frac{e \log \alpha_0}{2}\right)^{2n}\right)$$

or, since $\pi_1 < 1$:

$$E_n > \pi_1 \pi_2 \cdots \pi_n \left(1 - \log \alpha_0 - \frac{\alpha_0\left(\frac{e \log \alpha_0}{2}\right)^4}{1 - \alpha_0\left(\frac{e \log \alpha_0}{2}\right)^2}\right)$$

whenever $\alpha_0\left(\frac{e \log \alpha_0}{2}\right)^2 < 1$.

Choose particularly

$$\alpha = \frac{3}{2} \quad \text{and} \quad \alpha_0 = 1.51 \quad .$$

We obtain

$$E_n > 0.3(1 - \frac{1}{p_1}) \cdots (1 - \frac{1}{p_r}) \tag{19}$$

We study the number (R) of terms in E_n, on forming the following product

$$(1 - \frac{1}{p_1} - \cdots - \frac{1}{p_r})\ (1 - \frac{1}{p_1} - \cdots - \frac{1}{p_{t-1}})^2 \cdots (1 - \frac{1}{p_1} - \cdots - \frac{1}{p_{w-1}})^2 .$$

This product contains all the terms of E_n and more. The number $(r+1)$ of terms in the first factor is less than p_r, and in the second less than $p_r^{1/\alpha}$ etc. We obtain the number of terms of the product, on substituting all the terms $-\frac{1}{p}$ by $+1$, whence we conclude

$$R < p_r \cdot p_r^{2/\alpha} \cdots p_r^{2/\alpha^n} < p_r^{(\alpha+1)/(\alpha-1)} = p_r^5 \quad .$$

We can then give (14') the following form

$$N(D, x, p_1,\dots,p_r) > \frac{x}{D}\,0.3\left(1-\frac{1}{p_1}\right)\dots\left(1-\frac{1}{p_r}\right) - p_r^5 . \quad (20)$$

This formula is valid for all successive prime numbers $p_1,\dots,p_r$ with $p_1 \geq p_e$, where p_e denotes a determinable prime number.

Suppose particularly $p_1 = p_{e+1}$, the (e+1)-th prime number.

When the question is to calculate $N(D, x, 2,\dots,p_e, p_1,\dots,p_r)$, we can subjoin to our diagram (under (14)) the following:

$$\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{p_e}$$

$$\dots\dots$$

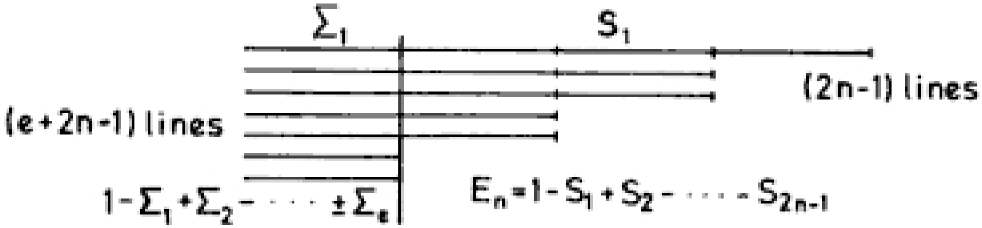
$$\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{p_e}$$

which gives rise to the expression

$$\left(1-\frac{1}{2}\right)\dots\left(1-\frac{1}{p_e}\right) = 1-\Sigma_1+\Sigma_2-\dots\pm\Sigma_e$$

containing 2^e terms, whenever the number of the lines are $\geq e$.

We obtain then the new diagram



giving rise to the new expression E_{n+1}:

$$\begin{aligned} E_{n+1} = 1 &- (\Sigma_1+S_1) + (\Sigma_2+S_1\Sigma_1+S_2) - \dots \\ &+ (\Sigma_e+S_1\,\Sigma_{e-1}+\dots+S_e) \\ &- (S_1\Sigma_e + S_2\,\Sigma_{e-1}+\dots+S_{e+1}) + \dots \\ &+ (S_{2n-e}\,\Sigma_e+\dots+S_{2n-1}\,\Sigma_1) + \dots + (S_{2n-1}\,\Sigma_e) \end{aligned}$$

or

$$E_{n+1} = (1 - \textstyle\sum_1 + \sum_2 - \ldots + \sum_e)\ (1 - S_1 + S_2 - \ldots + S_{2n-1})$$

$$= (1 - \frac{1}{2}) \ldots (1 - \frac{1}{p_e})E_n \quad ,$$

where we have supposed e to be even.

We obtain then by means of (19) the formula

$$N(D, x, 2, 3, \ldots, p_r) > \frac{x}{D}\, 0.3\ (1 - \frac{1}{2})\ (1 - \frac{1}{3}) \ldots (1 - \frac{1}{p_r}) - 2^e p_r^5 \tag{21}$$

valid for all $r > e$, where e denotes a determinable number, on noting that every term of $(1 - \frac{1}{2})(1 - \frac{1}{3}) \ldots (1 - \frac{1}{p_r})$ is multiplied by every term of E_n.

But in that case we can determine, by the Mertens' formula, a number c in a way that

$$N(D, x, 2, 3, \ldots, p_r) > \frac{0.168\, x}{D \log p_r} - 2^e p_r^5 \tag{22}$$

for all $r > c$, where c denotes a determinable number ($c \geq e$).

If we choose $D = 1$ and $P_r = p(\sqrt[6]{x})$, i.e.; the greatest prime number not exceeding $\sqrt[6]{x}$: $p_r \leq \sqrt[6]{x} < p_{r+1}$, we obtain particularly:

$$N\Big(1, x, 2, 3, \ldots, p(\sqrt[6]{x})\Big) > \frac{1.008\, x}{\log x} - 2^e\, x^{5/6} > \frac{x}{\log x}$$

for all $x > x_0$.

We can then state the following theorem:

When we efface from x consecutive numbers the terms from two to two, then from three to three, etc; finally from $p(\sqrt[6]{x})$ to $p(\sqrt[6]{x})$, there remain always more than $\frac{x}{\log x}$ terms, provided $x > x_0$.

The starting points of the effacements can be chosen as one would have it. x_0 denotes a determinable number.

We can also deduce, by means of the formula (22), the following theorem:

There exists always a number between n and $n+\sqrt{n}$, whose number of prime factors does not exceed eleven whenever $n > n_0$.

Choose in the formula (22)

$$D = 1, \quad x = \sqrt{n} \quad \text{and} \quad p_r = p(n^{1/11}) \;.$$

We obtain then

$$N(1, \sqrt{n}, 2,3,\ldots,p(n^{1/11})) > \frac{1.8\sqrt{n}}{\log n} - 2^e n^{5/11} > 1$$

for all $n > n_0$.

When we efface in the interval $[n, n+\sqrt{n}]$ all the multiples of two, three, etc. up to $p(n^{1/11})$, there remains therefore at least one number. We choose n as a starting point of the effacements. The numbers not effaced cannot be composed of 12 or more prime factors, because in that case one of these factors would be less than $\sqrt[12]{n+\sqrt{n}}$, and therefore less than $\sqrt[11]{n}$ for all $n > n_0$. But all these numbers being divisible by 2, 3,..., or $p(n^{1/11})$ are effaced.

§5. We have supposed that

$$2,\ 3,\ldots,p_r$$

in the formula (21) are successive prime numbers.

We generalize easily on studying the non-successive prime numbers

$$q_1, q_2, \ldots, q_{\alpha-1}, q_{\alpha+1}, \ldots, q_{\gamma-1}, q_{\gamma+1}, \ldots, q_r$$

forming a part of the successive prime numbers

$$q_1, q_2, \ldots, q_{\alpha-1}, q_\alpha, q_{\alpha+1}, \ldots, q_{\gamma-1}, q_\gamma, q_{\gamma+1}, \ldots, q_r \;,$$

where $q_1 = 2$ etc;

and we obtain as before (see (21)):

$$N(D,\ x,\ q_1,\dots,q_{\alpha-1},\ q_{\alpha+1},\dots,q_r)$$

$$> \frac{x}{D}\cdot 0.3\left(1-\frac{1}{q_1}\right)\ \dots\ \left(1-\frac{1}{q_{\alpha-1}}\right)\left(1-\frac{1}{q_{\alpha+1}}\right)\ \dots\ \left(1-\frac{1}{q_r}\right) - 2^e q_r^5$$

or

$$N(D,\ x,\ q_1,\dots,q_{\alpha-1},\ q_{\alpha+1},\dots,q_r)$$

$$> \frac{x}{D}\cdot 0.3\ \frac{\left(1-\frac{1}{q_1}\right)\ \dots\ \left(1-\frac{1}{q_r}\right)}{\left(1-\frac{1}{q_\alpha}\right)\ \dots\ \left(1-\frac{1}{q_\gamma}\right)} - 2^e q_r^5 \quad .$$

Hence we conclude

$$N(D,\ x,\ q_1,\dots,q_{\alpha-1},\ q_{\alpha+1},\dots,q_r)$$

$$> \frac{0.168\,x}{D\log q_r}\cdot\frac{1}{\left(1-\frac{1}{q_\alpha}\right)\ \dots\ \left(1-\frac{1}{q_\gamma}\right)} - 2^e q_r^5 \quad .$$

We study now an arithmetical progression extended from 0 to x:

$$\Delta \qquad \Delta+D \qquad \Delta+2D\ \dots\dots \quad ,$$

Δ and D being relatively prime. Suppose

$$D = q_\alpha^a\ \dots\ q_\gamma^c \quad .$$

We efface now the numbers being divisible by

$$q_1,\dots,q_{\alpha-1},\ q_{\alpha+1},\dots,q_{\gamma-1},\ q_{\gamma+1},\dots,q_r$$

on choosing $q_r = q(\sqrt[6]{x})$. We obtain

$$N(D,\ x,\ q_1,\dots,q_{\alpha-1},\ q_{\alpha+1},\dots,q_r) > \frac{0.168\,x}{\phi(D)\log q_r} - 2^e q_r^5$$

$$> \frac{1.008\,x}{\phi(D)\log x} - 2^e\, x^{5/6} > \frac{1}{\phi(D)}\cdot\frac{x}{\log x}$$

for all $x > x_0$.

The numbers not effaced are indivisible by

$$q_1,\dots,q_{\alpha-1},\ q_{\alpha+1},\dots,q_{\gamma-1},\ q_{\gamma+1},\dots,q_r$$

but they are also indivisible by

$$q_\alpha,\dots,q_r \quad ,$$

since Δ and D are relatively prime. The numbers not effaced contain therefore five or less prime factors.

Hence we deduce the following theorem analogous to that of Dirichlet:

Every arithmetical progression, whose first term and difference are relatively prime, contains an infinity of terms, whose number of prime factors does not exceed five.

§6. Now we study the Merlin's sieve, where one efface double all the multiples of three, five, etc. up to p_r. On generalizing, we study the following arithmetical progression

Δ	$\Delta + D$	$\Delta + 2D$	...
a_1	$a_1 + p_1$	$a_1 + 2p_1$	...
b_1	$b_1 + p_1$	$b_1 + 2p_1$	...
	...	...	
a_r	$a_r + p_r$	$a_r + 2p_r$	...
b_r	$b_r + p_r$	$b_r + 2p_r$	...

All the letters are defined in §2. Moreover we suppose $a_i \neq b_i$ and $p_1 \geq 3$. Denote by

$$P(\Delta,\ D,\ x,\ a_1,\ b_1,\ p_1,\dots,a_r,\ b_r,\ p_r)$$

or more briefly by

$$P(D,\ x,\ p_1,\dots,p_r)$$

the number of the terms of the first progression, which are different from all the terms of the other progressions. We deduce as before the fundamental formula

$$P(\Delta,\ x,\ a_1,\ b_1,\ p_1,\dots,a_r,\ b_r,\ p_r)$$

$$= P(\Delta,\ D,\ x,\ a_1,\ b_1,\ p_1,\dots,a_{r-1},\ b_{r-1},\ p_{r-1})$$

$$- P(\Delta',\ Dp_r,\ x,\ a_1,\ b_1,\ p_1,\dots,a_{r-1},\ b_{r-1},\ p_{r-1})$$

$$- P(\Delta'',\ Dp_r,\ x,\ a_1,\ b_1,\ p_1,\dots,a_{r-1},\ b_{r-1},\ p_{r-1}) \quad ,$$

or more briefly

$$P(D,\ x,\ p_1,\dots,p_r) = P(D,\ x,\ p_1,\dots,p_{r-1})$$

$$- 2P(Dp_r,\ x,\ p_1,\dots,p_{r-1}) \quad . \qquad (23)$$

It can give rise to no misunderstanding, since we have written $2P(Dp_r,\ x,\ p_1,\dots,p_{r-1})$ when one remembers that it denotes a sum of two expressions of the form $P(\Delta,\ Dp_r,\ x,\ a_1,\ b_1,\ p_1,\dots,a_{r-1},\ b_{r-1},\ p_{r-1})$.

We obtain as before, by means (23), the general formula analogous to (5)

$$\frac{D}{x} P(D,\ x,\ p_1,\dots,p_r) > 1 - \sum_{a \le r} \frac{2}{p_a} + \sum_{\omega_1}\sum \frac{2^2}{p_a p_b}\left(1 - \sum_{c<b} \frac{2}{p_c}\right)$$

$$+ \sum_{\omega_1'}\sum\ \sum_{\omega_1}\sum \frac{2^4}{p_a p_b p_c p_d}\left(1 - \sum_{e<d} \frac{2}{p_e}\right) + \dots + \frac{RD}{x} \quad , \qquad (24)$$

where $\omega_1' \le \omega_1$ etc.

R denotes the number of the terms of the form $\pm \frac{1}{n}$ in the formula, (where $\frac{2}{n} = \frac{1}{n} + \frac{1}{n}$, etc.). We have supposed that $p_1 \ge 3$. Besides the designations, all are the same as in the formula (5).

We can also give the formula (24) the following form, on supposing particularly $p_1 = 3$, $p_2 = 5$, $p_3 = 7$, etc.:

$$P(D,\ x,\ 3,\ 5,\dots,p_r) > \frac{x}{D}\left[1-\frac{2}{3}-\frac{2}{5}-\dots-\frac{2}{p_r}\right.$$

$$+\frac{4}{5\cdot 3}+\frac{4}{7\cdot 3}+\frac{4}{7\cdot 5}\left(1-\frac{2}{3}\right)+\dots+\frac{4}{p_r\cdot 3}+\frac{4}{p_r\cdot 5}\left(1-\frac{2}{3}\right)$$

$$\left.+\frac{1}{p_r\cdot 7}\begin{pmatrix}1-\frac{2}{3}-\frac{2}{5}\\ \\ +\frac{4}{5\cdot 3}\end{pmatrix}+\frac{4}{p_r\cdot 7}\begin{pmatrix}1-\frac{2}{3}-\frac{2}{5}-\frac{2}{7}\\ +\frac{4}{5\cdot 3}\\ +\frac{4}{7\cdot 3}+\frac{4}{7\cdot 5}\left(1-\frac{2}{3}\right)\end{pmatrix}+\dots\right]-R\ , \tag{25}$$

where one can set aside every term, (the subsequent parenthesis included), which follows the sign + .

We give an example, one studying the following arithmetical progression extended from 0 to 11,776

1	3	5	7	9	11	13	15	...	11,769	11,771	11,773	11,775	
0	3	6		9	12		15	...	11,769	11,772		11,775	
1	4		7	10		13			11,770		11,773		11,776
	. . .												
0							19	...	11,761				
						15		...	11,757				11,776

The starting points of the effacements are 0 and 11,776 (see §1).

We obtain by means of (25), on observing that $a_i \neq b_i$, since $11{,}776 = 2^9\cdot 23$ is indivisible by 3, 5, 7,...,19 :

$$P(2,\ 11{,}776,\ 3,\ 5,\ldots,19) > \frac{11{,}776}{2}\left[1-\frac{2}{3}-\frac{2}{5}-\frac{2}{7}-\frac{2}{11}-\frac{2}{13}-\frac{2}{17}\right.$$

$$-\frac{2}{19}+\frac{4}{5\cdot 3}+\frac{4}{7\cdot 3}+\frac{4}{7\cdot 5}\left(1-\frac{2}{3}\right)+\frac{4}{11\cdot 3}+\frac{4}{11\cdot 5}\left(1-\frac{2}{3}\right)$$

$$+\frac{4}{11\cdot 7}\begin{pmatrix}1-\frac{2}{3}-\frac{2}{5}\\ +\frac{4}{5\cdot 3}\end{pmatrix}+\frac{4}{13\cdot 3}+\frac{4}{13\cdot 5}\left(1-\frac{2}{3}\right)+\frac{4}{13\cdot 7}\begin{pmatrix}1-\frac{2}{3}-\frac{2}{5}\\ +\frac{4}{5\cdot 3}\end{pmatrix}$$

$$\left.+\frac{4}{17\cdot 3}+\frac{4}{17\cdot 5}\left(1-\frac{2}{3}\right)+\frac{4}{19\cdot 3}+\frac{4}{19\cdot 5}\left(1-\frac{2}{3}\right)\right]-R\ ,$$

where

$$R = 1 + 14 + 4 + 16 + 52 + 52 + 32 = 171\ ,$$

whence $P(2,\ 11{,}776,\ 3,\ 5,\ldots,19) > 296 - 171 = 125$.

The number (t) not effaced of the first progression, whose number is more than 125, having the following property: t and $11{,}776-t$ are indivisible by 2, 3, 5,...,19. They cannot composed of three or more prime factors, because otherwise one of these factors would be less than $\sqrt[3]{11{,}776} < 22.9$.

One can then write the number 11,776 as the sum of two numbers, whose number of prime factors do not exceed 2, in 125 or more different ways.

However, I have not succeeded in giving an example of the justness of the theorem of Goldbach by this method.

Nevertheless we see that we can deduce important results by means of the formula (24), the method being completely analogous to that employed above.

One should only replace $\frac{1}{p_i}$ by $\frac{2}{p_i}$ everywhere.

We calculate by means of the same diagram in the form of stairs as in §4 on replacing $\frac{1}{p_i}$ by $\frac{2}{p_i}$. One should then replace the sums and the products considered in §4 by the following:

$$\sigma_1 = \frac{2}{p_t} + \ldots + \frac{2}{p_r} < 2\log\alpha_0 \quad , \quad \text{etc.} \; ,$$

and

$$\Pi_1 = \left(1 - \frac{2}{p_t}\right) \ldots \left(1 - \frac{2}{p_r}\right) < \frac{1}{\alpha_0^2} \quad , \quad \text{etc.} \quad ,$$

on applying the following formula

$$\prod_3^x \left(1 - \frac{2}{p}\right) = \frac{0.8322}{\log^2 x} \cdot e^{c\theta/\log x} \quad .$$

We suppose now $2\log\alpha_0 < 1$.

We deduce the following formula analogous to (18):

$$E_{m+1} > \Pi_{m+1} E_m - (e \log \alpha_0)^{2m+2} \quad ,$$

whence one gets

$$E_n > \Pi_1 \ldots \Pi_n \left(1 - 2\log\alpha_0 - \frac{\alpha_0^2 (e \log \alpha_0)^4}{1 - \alpha_0^2 (e \log \alpha_0)^2}\right) \quad .$$

Choose particularly

$$\alpha = \frac{5}{4} = 1.25 \quad \text{and} \quad \alpha_0 = 1.2501 \quad .$$

We obtain then

$$E_n > 0.05\left(1 - \frac{2}{p_1}\right) \ldots \left(1 - \frac{2}{p_r}\right) \quad . \tag{26}$$

We study the number (R) of terms in E_n, on forming the following product

$$\left(1 - \frac{2}{p_p} - \ldots - \frac{2}{p_r}\right)\left(1 - \frac{2}{p_1} - \ldots - \frac{2}{p_{t-1}}\right)^2 \ldots \left(1 - \frac{2}{p_1} - \ldots - \frac{2}{p_{w-1}}\right)^2 .$$

This product contains all the terms of E_n and more. The number $(2r + 1)$ of terms in the first factor is less than p_r whenever $p_1 > 3$, and in the second less than $p_r^{1/\alpha}$, etc. Hence we conclude

$$R < p_r \, p_r^{2/\alpha} \ldots p_r^{2/\alpha^n} < p_r^{(\alpha+1)/(\alpha-1)} = p_r^9 \quad .$$

We obtain then the formula

$$P(D,\ x,\ p_1,\dots,p_r) > \frac{x}{D} \cdot 0.05\left(1-\frac{2}{p_1}\right) \dots \left(1-\frac{2}{p_r}\right) - p_r^9 \tag{27}$$

a formula which is valid for all successive prime numbers $p_1,\dots,p_r$, whenever $p_1 \geq p_e$, where p_e denotes a determinable prime number.

We obtain also a formula analogous to (21):

$$P(D,\ x,\ 3,\ 5,\dots,p_r) > \frac{x}{D} \cdot 0.05\left(1-\frac{2}{3}\right) \dots \left(1-\frac{2}{p_r}\right) - 3^3 p_r^9 \tag{28}$$

valid for all $r > e$.

Hence we conclude

$$P(D,\ x,\ 3,\ 5,\dots,p_r) > \frac{x}{D} \cdot \frac{0.041}{(\log p_r)^2} - e^e p_r^9 \tag{29}$$

for all $r > c$, where $c \geq e$.

Choose particularly $p_r = p(x^{1/10})$. We obtain then

$$P\left(D,\ x,\ 3,\ 5,\dots,p(x^{1/10})\right) > \frac{0.41\,x}{D(\log x)^2} - 3^e\ x^{9/10} > \frac{0.4\,x}{D(\log x)^2} \tag{30}$$

for all $x > x_0$.

On supposing $D = 1$, we can therefore state the following theorem:

When we efface double among x terms all the multiples of three, five, etc. up to $p(x^{1/10})$, there always remain more than $\dfrac{0.4\,x}{(\log x)^2}$ terms provided $x > x_0$.

We have supposed that

$$a_i \neq b_i \quad .$$

that is to say that none of the double effacements are reduced to a single one. When the question is to determine the Goldbachian partitions of the number $x = 2^s\ p_\alpha^t \dots p_\gamma^v$, one see yet that

$$a_\alpha = b_\alpha,\dots,a_\gamma = b_\gamma \quad .$$

But the lower bound for P will not naturally less, when one reduces the effacements (compare §5). One should then replace $\frac{2}{p_\alpha}$ by $\frac{1}{p_\alpha}$ and $\frac{2}{p_\gamma}$ by $\frac{1}{p_\gamma}$. We obtain then the new lower bound for P:

$$\frac{0.4\,x}{D(\log x)^2} \cdot \frac{\left(1-\frac{1}{p_\alpha}\right)\ \ldots\ \left(1-\frac{1}{p_\gamma}\right)}{\left(1-\frac{2}{p_\alpha}\right)\ \ldots\ \left(1-\frac{2}{p_\gamma}\right)} > \frac{0.4\,x}{D(\log x)^2} \quad .$$

Hence we conclude, as in the previous example, on choosing $D = 2$, the following theorem, analogous to that of Goldbach:

One can write even number x, greater than x_0, as a sum of two numbers, whose numbers of prime factors do not exceed nine. x_0 denotes a determinable number and the prime factors can be different or not.

We can also deduce the following theorem:

There exists an infinity of the pairs of numbers, having the difference 2, in the class of the numbers whose numbers of prime factors do not exceed nine.

§7. We can also determine an upper bound for the number of numbers, which remain non-effaced on employing the sieves of Eratosthenes and Merlin.

We apply the following inequality

$$N(\Delta, D, x, a_1, p_1, \ldots, a_r, p_r, \ldots, a_n, p_n)$$

$$\leq N(\Delta, D, x, a_1, p_1, \ldots, a_r, p_r)$$

or more briefly

$$N(D, x, p_1, \ldots, p_r, \ldots, p_n) \leq N(D, x, p_1, \ldots, p_r) \quad . \qquad (31)$$

where $r < n$.

We apply also the formula

$$N(D,\ x,\ p_1,\dots,p_r) = N(D,\ x) - \sum_{a \le r} N(Dp_a,\ x)$$

$$+ \sum_{a \le r} \ \sum_{b < a} N(Dp_a p_b,\ p_1,\dots,p_{b-1}) \ . \qquad (3')$$

To estimate the terms of the last sum, we apply (31) and the same formula (3'). On continuing we obtain the formula analogous to (14):

$$N(D,\ x,\ p_1,\dots,p_r) < \frac{x}{D}\left[1 - \sum_{a \le r} \frac{1}{p_a} + \sum_{a \le r} \ \sum_{\substack{b < a \\ b < r}} \frac{1}{p_a p_b}\right.$$

$$- \sum_{a \le r} \ \sum_{\substack{b < a \\ b < r}} \ \sum_{\substack{c < b \\ c < t}} \frac{1}{p_a p_b p_c}$$

$$\left. + \sum_{a \le r} \ \sum_{\substack{b < a \\ b < r}} \ \sum_{\substack{c < b \\ c < t}} \ \sum_{\substack{d < c \\ d < t}} \frac{1}{p_a p_b p_c p_d} - \dots \right] + R\ , \qquad (32)$$

or more briefly

$$D(D,\ x,\ p_1,\dots,p_r) < \frac{x}{D}\left[1 - S_1 + S_2 - \dots + S_{2n}\right] + R\ ,$$

where the expression

$$E_n = 1 - S_1 + S_2 - \dots + S_{2n}$$

is calculated by means of the diagram

On employing the same method as before, we obtain

$$E_{m+1} < \Pi_{m+1} E_m + \left(\frac{e \log \alpha_0}{2}\right)^{2m+3}$$

and particularly

$$E_1 < \Pi_1 + \left(\frac{e \log \alpha_0}{2}\right)^3$$

whence

$$E_2 < \Pi_1\Pi_2 \left[1 + \alpha_0\left(\frac{e \log \alpha_0}{2}\right)^3 + \alpha_0^2\left(\frac{e \log \alpha_0}{2}\right)^5\right] .$$

On continuing we obtain at last

$$E_n < \Pi_1 \ldots \Pi_n \left[1 + \alpha_0\left(\frac{e \log \alpha_0}{2}\right)^3 + \alpha_0^2\left(\frac{e \log \alpha_0}{2}\right)^5 + \ldots\right]$$

or

$$E_n < (1 - \frac{1}{p_1}) \ldots (1 - \frac{1}{p_r}) \left(1 + \frac{\alpha_0\left(\frac{e \log \alpha_0}{2}\right)^3}{1 - \alpha_0\left(\frac{e \log \alpha_0}{2}\right)^2}\right), \tag{33}$$

whenever $\alpha_0\left(\frac{e \log \alpha_0}{2}\right)^2 < 1$.

Choose particularly

$$\alpha = \frac{3}{2} \quad \text{and} \quad \alpha_0 = 1.51 \quad .$$

We obtain then

$$E_n < 1.505 \, (1 - \frac{1}{p_1}) \ldots (1 - \frac{1}{p_r}) \quad .$$

We study the number (R) of terms in E_n on forming the following product

$$\left(1 - \frac{1}{p_1} - \ldots - \frac{1}{p_r}\right)^2 \left(1 - \frac{1}{p_1} - \ldots - \frac{1}{p_{t-1}}\right)^2 \cdots \left(1 - \frac{1}{p_1} - \ldots - \frac{1}{p_{w-1}}\right)^2 \quad .$$

We see, as before, that

$$R < p_r^2\, p_r^{2/\alpha} \ldots p_r^{2/\alpha^n} < p_r^{(2\alpha/\alpha-1)} = p_r^6 \ .$$

We can give (32) the following form

$$N(D,\ x,\ p_1,\ldots,p_r) < \frac{x}{D} \cdot 1.505\left(1-\frac{1}{p_1}\right)\ldots\left(1-\frac{1}{p_r}\right) + p_r^6 \ .$$

Thence we conclude the formula

$$N(D,\ x,\ 2,\ 3,\ldots,p_r)$$
$$< \frac{x}{D} \cdot 1.505\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{p_r}\right) + 2^e p_r^6$$

valid for all $r > e$.

But in virtue of the Mertens' formula we obtain

$$N(D,\ x,\ 2,\ 3,\ldots,p_r) < \frac{0.9\,x}{D \log p_r} + 2^e p_r^6$$

for all $r > c$, where $c \geq e$.

Choose particularly $p_r = p(2\sqrt[7]{x})$. Thence we conclude that

$$\sqrt[7]{x} < p_r \leq 2\sqrt[7]{x}$$

on applying a celebrated theorem of Tchebycheff.

Therefore we obtain

$$N(1,\ x,\ 2,\ 3,\ldots,p(2\sqrt[7]{x})) < \frac{6.5\,x}{\log x} + 2^{e+6}\,x^{6/7} < \frac{7x}{\log x} \tag{34}$$

for all $x > x_0$.

On applying the inequality (31), we obtain

$$N(1,\ x,\ 2,\ldots,p(\sqrt{x})) \leq N(1,\ x,\ 2,\ldots,p(\sqrt[6]{x}))$$
$$\leq N(1,\ x,\ 2,\ldots,p(2\sqrt[7]{x})) < \frac{7x}{\log x}$$

for all $x > x_0$.

Thence we conclude particularly that

$$\pi(x) - \pi(\sqrt{x}) + 1 < \frac{7x}{\log x} .$$

whence

$$\pi(x) < \frac{7x}{\log x} + \sqrt{x} < \frac{8x}{\log x}$$

for all $x > x_0$, $\pi(x)$ denoting the number of the prime numbers not exceeding x.

On comparing the theorem in §4, we obtain also

$$\frac{x}{\log x} < N(1, x, 2, \ldots, p(\sqrt[6]{x})) < \frac{7x}{\log x} . \tag{35}$$

When we efface among x terms all the multiples of two, three etc. up to $p(\sqrt[6]{x})$, there remain always N terms, where N is a number lying in the interval $\left[\frac{x}{\log x}, \frac{7x}{\log x}\right]$, whenever $x > x_0$.

We study at last the sieve of Merlin. We obtain the formula analogous to (33):

$$E_n < (1 - \frac{2}{p_1}) \ldots (1 - \frac{2}{p_r}) \left(1 + \frac{\alpha_0^2 (e \log \alpha_0)^3}{1 - \alpha_0^2 (e \log \alpha_0)^2}\right)$$

Choose particularly

$$\alpha = 1.25 \quad \text{and} \quad \alpha_0 = 1.2501 ,$$

whence one gets

$$E_n < 1.82 (1 - \frac{2}{p_1}) \ldots (1 - \frac{2}{p_r}) .$$

Thence we deduce as before

$$P(D, x, 3, 5, \ldots, p_r)$$

$$< \frac{x}{D} \cdot 1.82 (1 - \frac{2}{3})(1 - \frac{2}{5}) \ldots (1 - \frac{2}{p_r}) + 3^e p_r^{10}$$

or

$$P(D, x, 3, 5, \ldots, p_r) < \frac{1.6 x}{D(\log p_r)^2} + 3^e p_r^{10} . \tag{36}$$

for all $r > c$, where $c \geq e$ (see §6).

Choose now $p_r = p(2x^{1/11})$. We obtain then

$$P(D,\ x,\ 2,\ 3,\dots,p(2x^{1/11})$$

$$< \frac{194\,x}{D(\log x)^2} + 3^{e+10}x^{10/11} < \frac{195\,x}{D(\log x)^2}$$

for all $x > x_0$.

We apply now the inequality

$$P(D,\ x,\ 2,\ 3,\dots,p(\sqrt{x})) \leq P(D,\ x,\ 2,\ 3,\dots,p(2\sqrt[11]{x}))$$

and the equation

$$Z(x) - Z(\sqrt{x} + 2) + 1 = P(2,\ x,\ 2,\ 3,\dots,p(\sqrt{x})) \quad ,$$

where $Z(x)$ denotes the number of the twin prime numbers not exceeding x, and where we have chosen 0 and 2 as starting points of the effacements.

We obtain therefore

$$Z(x) < \frac{195\,x}{2(\log x)^2} + \sqrt{x} + 2$$

or

$$Z(x) < \frac{100\,x}{(\log x)^2}$$

for all $x > x_0$, where x_0 denotes a determinable number. Here $Z(x)$ denotes the number of the twin prime numbers not exceeding x.

(See: Skr. Norske Vid.-Akad; Kristiania, I (1920) no. 3. Some formulas in the text are slightly modified by the Editor).

Translated by Yu Kun Rui