First, hamon toda for the wonderful gift: the three issues of the AAM for my 65-th birthday.

Let $P(n)$ denote the set of the partitions of $n$. As usual, for $\lambda \in P(n)$, $\ell(\lambda)$ is the number of parts of $\lambda$. Let $SP(n) \subset P(n)$ denote the partitions of $n$ of odd distinct parts. For example, $SP(8) = \{(7,1), (5,3)\}$.

**Fact:**
1. $|P(n) \setminus SP(n)|$ is always even, and moreover,
2. for exactly half of the partitions $\lambda \in P(n) \setminus SP(n)$, $\ell(\lambda)$ is even.

I can prove this purely combinatorial fact using the character theory of $S_n$ and $A_n$. Is there a direct - purely combinatorial - proof?

Here is a "bijective" proof that $|P(n) \setminus SP(n)|$ is always even:

Let $P_{sym}(n) = \{\lambda \vdash n \mid \lambda = \lambda'\}$, so $p_{sym}(n) = |P_{sym}(n)|$. Let $\lambda = \lambda'$, write it in the Frobenius notation $\lambda = (a_1, \ldots, a_k \mid a_1, \ldots, a_k)$ then map it

$$h : (a_1, \ldots, a_k \mid a_1, \ldots, a_k) \to (2a_1 + 1, \ldots, 2a_k + 1) \in SP(n),$$

then $h : P_{sym}(n) \to SP(n)$ is a bijection, so $|P_{sym}(n)| = |SP(n)|$. This yields a bijection between $P(n) \setminus SP(n)$ and $P(n) \setminus P_{sym}(n)$. But $P(n) \setminus P_{sym}(n) = \{\lambda \vdash n \mid \lambda \neq \lambda'\}$ is a union of the pairs $\lambda, \lambda'$, hence $|P(n) \setminus P_{sym}(n)| = |P(n) \setminus SP(n)|$ is even. We would get a proof of 2. if we can extend the bijection $h : P_{sym}(n) \to SP(n)$ into a bijection $h^* : P(n) \to P(n)$ – therefore also $h^* : P(n) \setminus P_{sym}(n) \to P(n) \setminus SP(n)$ – such that to each pair $\lambda, \lambda'$, exactly one partition of the pair $h^*(\lambda), h^*(\lambda')$ is of even length So, the question is how to find $h^*$ for all $n$.

lehitraot,

Amitai