Experimental Math Project

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1 Introduction

Let the general form of the recurrence $\{a(n)\}_{n\in\mathbb{N}}$ be defined by

$$a(n)a(n-k) = a(n-i)a(n-k+i) + a(n-j) + a(n-k+j) \qquad (1)$$

with initial conditions $a(n) = 1$ for all $n \leq k$. To be clear as to the values of (k, i, j) I may label the sequence by $\{a_{k,i,j}(n)\}$. For certain choices of (k, i, j) this recurrence relation is conjectured to generate an infinite sequence of integers.

Conjecture 1. *Consider the quadratic recurrence (1).*

- *In the case where k is even:*
  - *If i is odd, then $j = \frac{k}{2}$ defines a recurrence that generates only integers.*
  - *If i is even, then $j = \frac{i}{2}$, $j = \frac{k}{2}$, and $j = \frac{k-i}{2}$ define recurrences that generate only integers.*

- *In the case where k is odd:*
  - *If i is odd then $j = \frac{k-i}{2}$ defines a recurrence that generates only integers.*
  - *If i is even then $j = \frac{i}{2}$ defines a recurrence that generates only integers.*

Furthermore, all other values of j do not define a recurrence that generates integers exclusively.

In my previous paper, coauthored with Paul Heideman, we proved the following:

Proposition 1. *The sequence* $\{a_{2K+1,1,K}(n)\} = \{a(n)\}$ *defined by the quadratic recurrence relation (1), with initial conditions* $a(n) = 1$ *for* $n \leq 2K+1$ *is also generated by the linear recurrence relation defined by*

$$b(n) = \left[2K^2 + 8K + 4\right](b(n-2K) - b(n-4K)) + b(n-6K) \quad (2)$$

with the first $6K+1$ *terms taken to be the first* $6K+1$ *terms of the sequence generated by the quadratic recurrence.*

We also proved that the first $6K+1$ terms are defined by a piecewise polynomial in K and n, thus proving that the terms are all integers. After showing this I had hope that the proof could be generalized and more integrality results could be proven about the general form of the quadratic recurrence.

2 New Results

Using Maple I began to experiment, hoping to find linear recurrences for the sequences generated by the quadratic recurrence (1) with $i \neq 1$. I began by looking at the sequences for patterns. The sequences that were most interesting to me were the ones of the form $\{...n_1, n_1, ..., n_1, n_2, n_2, ..., n_2, n_3, n_3, ..., n_3, ...\}$ where each n_l is repeated the same number of times. I noticed a pattern in the (k, i, j) used to generate the sequence from (1). I then conjectured, and subsequently proved, the following proposition.

Proposition 2. *Consider the sequence* $\{a_{(2K+1)i,i,Ki}(n)\}_{n\in\mathbb{N}}$ *from (1), i.e.-* $\{a_{(2K+1)i,i,Ki}(n)\} = \{a(n)\}$ *is defined by:*

$$a(n)a(n-(2K+1)i) = a(n-i)a(n-2Ki) + a(n-Ki) + a(n-(K+1)i) \quad (3)$$

with initial conditions $a(n) = 1$ *for* $n \leq (2K+1)i$. *Consider also the sequence* $\{a_{2K+1,1,K}(n)\} = \{c(n)\}$ *defined by*

$$c(n)c(n-(2K+1)) = c(n-1)c(n-2K) + c(n-K) + c(n-(K+1)) \quad (4)$$

with initial conditions $c(n) = 1$ *for* $n \leq 2K+1$. *Then we have the following relationship*

$$a(Li+m) = c(L+1) \text{ for } m \in [1, i] \quad (5)$$

Proof. I will prove this by induction on L. The base case is to verify the relationship for $L = 0, 1, 2, ..., 2K + 1$.

- L=0,1,...,2K: In these cases we have that $Li + m \in [1, 2Ki + i] = [1, (2K + 1)i]$, and therefore, $a(Li + m) = 1 = c(L + 1)$.
- L=2K+1: In this case we must use the definition of the recurrence for $a(n)$.

$$a((2K+1)i+m)a(m) = a(2Ki+m)a(i+m)+a((K+1)i+m)+a(Ki+m)$$

$$a((2K + 1)i + m) \cdot 1 = 1 \cdot 1 + 1 + 1$$
$$a((2K + 1)i + m) = 3 = c(2K + 2)$$

Now for the inductive step. Assume, as the inductive hypothesis, that $a(Mi + m) = c(M + 1)$ for $M < L$. Then, let $L > 2K + 1$:

$$\begin{aligned}
a(Li+m)a(Li+m-(2K+1)i) &= a(Li+m-i)a(Li+m-2Ki)+\\
&\quad +a(Li+m-Ki)+a(Li+m-(K+1)i)\\
a(Li+m)a((L-2K-1)i+m) &= a((L-1)i+m)a((L-2K)i+m)+\\
&\quad +a((L-K)i+m)+a((L-K-1)i+m)\\
a(Li+m)c(L-2K) &= c(L)c(L-2K+1)c(L-K+1)+c(L-K)\\
a(Li+m)c((L+1)-(2K+1)) &= c((L+1)-1)c((L+1)-2K)+\\
&\quad c((L+1)-K)+c((L+1)-(K+1))
\end{aligned}$$

But we know that the sequence $\{c(n)\}$ satisfies (9), so this implies that $a(Li + m) = c(L + 1)$. Then by induction, $a(Li + m) = c(L + 1)$ for all $L \in \mathbb{N}$.

$\square$

I wrote a program to find a minimal linear recurrence for a given sequence. After some time I noticed a pattern in the conjectured linear recurrences: if $gcd(k, i, j) = 1$ then the recurrence would have 4 terms similar to (2); otherwise the recurrence had many more terms, but seemed to follow a pattern in the coefficients. From the experimental methods, I came up with the following proposition.

Proposition 3. *If the sequence* $\{a_{(2K+1)i,i,Ki}(n)\} = \{a(n)\}_{n\in\mathbb{N}}$ *is given by equation (3) then it is also annihilated by the linear recurrence relation defined by*

$$\begin{aligned} a(n) &= a(n-1) + \sum_{l_1=1}^{2K-1}\Big(-a(n-l_1 i) + a(n-(l_1 i+1))\Big) + \\ &+ \sum_{l_2=2K}^{4K-1}\big[2K^2+8K+3\big]\Big(a(n-l_2 i) - a(n-(l_2 i+1)\Big) + \\ &+ \sum_{l_3=4K}^{6K-1}\Big(-a(n-l_3 i) + a(n-(l_3 i+1))\Big) \end{aligned}$$

Proof. Define $\phi(n)$ by

$$\begin{aligned} \phi(n) &= a(n) - a(n-1) - \sum_{l_1=1}^{2K-1}\Big(-a(n-l_1 i) + a(n-(l_1 i+1))\Big) - \\ &- \sum_{l_2=2K}^{4K-1}\big[2K^2+8K+3\big]\Big(a(n-l_2 i) - a(n-(l_2 i+1)\Big) - \\ &- \sum_{l_3=4K}^{6K-1}\Big(-a(n-l_3 i) + a(n-(l_3 i+1))\Big) \end{aligned}$$

Any number n is congruent to $1, 2, ...,$ or i *mod* i, thus we can write $n = Li + m$ for some $L > 6K$ and $m \in \{1, 2, ..., i\}$. Then we can rewrite $\phi(n)$ as $\phi(Li + m)$:

$$\begin{aligned} \phi(Li+m) &= a(Li+m) - a(Li+(m-1)) - \\ &- \sum_{l_1=1}^{2K-1}\Big(-a((L-l_1)i+m) + a((L-l_1)i+(m-1))\Big) - \\ &- \sum_{l_2=2K}^{4K-1}\big[2K^2+8K+3\big]\Big(a((L-l_2)i+m) - a((L-l_2)i+(m-1))\Big) - \\ &- \sum_{l_3=4K}^{6K-1}\Big(-a((L-l_3)i+m) + a((L-l_3)i+(m-1))\Big) \end{aligned}$$

- Case 1: $(m \neq 1)$ We can now use Proposition 2 to simplify $\phi(Li + m)$

  $$\phi(Li+m) = c(L+1) - c(L+1) -$$

$$
\begin{aligned}
& - \sum_{l_1=1}^{2K-1} \Big(- c(L-l_1+1) + c(L-l_1+1) \Big) - \\
& - \sum_{l_2=2K}^{4K-1} \big[2K^2+8K+3\big] \Big(c(L-l_2+1) - c(L-l_2+1) \Big) - \\
& - \sum_{l_3=4K}^{6K-1} \Big(- c(L-l_3+1) + c(L-l_3+1) \Big) \\
\phi(Li+m) \;&=\; 0
\end{aligned}
$$

- Case 2: $(m = 1)$ In this case we will have to simplify $\phi(Li + m) = \phi(Li+1)$ slightly differently since we often subtract 1 from $(L-l_k)i+m$, and $a((L - l_k)i + m) \neq a((L - l_k)i + (m - 1))$ as it did in case 1.

$$
\begin{aligned}
\phi(Li+1) \;=\;& a(Li+1) - a((L-1)i+i) - \\
& - \sum_{l_1=1}^{2K-1} \Big(- a((L-l_1)i+1) + a((L-l_1-1)i+i) \Big) - \\
& - \sum_{l_2=2K}^{4K-1} \big[2K^2+8K+3\big] \Big(a((L-l_2)i+1) - a((L-l_2-1)i+i) \Big) - \\
& - \sum_{l_3=4K}^{6K-1} \Big(- a((L-l_3)i+1) + a((L-l_3-1)i+i) \Big)
\end{aligned}
$$

Again we can use Proposition 2 to write $\phi(Li + 1)$ in terms of the $\{c(n)\}$ sequence.

$$
\begin{aligned}
\phi(Li+1) \;=\;& c(L+1) - c(L) - \\
& - \sum_{l_1=1}^{2K-1} \Big(- c(L-l_1+1) + c(L-l_1) \Big) - \\
& - \sum_{l_2=2K}^{4K-1} \big[2K^2+8K+3\big] \Big(c(L-l_2+1) - c(L-l_2) \Big) - \\
& - \sum_{l_3=4K}^{6K-1} \Big(- c(L-l_3+1) + c(L-l_3) \Big)
\end{aligned}
$$

Each of these sums is telescoping, so we can write them as

$$
\sum_{l_1=1}^{2K-1} \Big(- c(L-l_1+1) + c(L-l_1) \Big) \;=\; -c(L) + c(L-(2K-1))
$$

$$\sum_{l_2=2K}^{4K-1}\left[2K^2+8K+3\right]\Big(c(L-l_2+1)-c(L-l_2)\Big) =$$

$$\left[2K^2+8K+3\right]\Big(c(L-2K+1) - c(L-(4K-1))\Big)$$

$$\sum_{l_3=4K}^{6K-1}\Big(-c(L-l_3+1)+c(L-l_3)\Big) =$$

$$-c(L-4K+1) + c(L-(6K-1))$$

So we can simplify $\phi(Li+1)$ as

$$\phi(Li+1)=c(L+1)-\left[2K^2+8K+4\right]\Big(c((L+1)-2K)+c((L+1)-4K)\Big)-c((L+1)-6K)$$

But we know from Proposition 1 that this equals 0.

In either case we have $\phi(Li+m)=0$ without using induction. Therefore, the linear recurrence in question annihilates the sequence generated by the quadratic recurrence (3) □

When I proved Proposition 2 and 3 I noticed they depended on the recurrence for k odd and $i=1$. I wondered if similar statements would be true for k even and $i=1$. The first step was to find if the case where k is even and $i=1$ had a nice linear recurrence. The following two propositions are the analog to what I proved in my paper with Paul Heideman for the case where k is even and $i=1$. Their proofs are almost identical to the proofs given in that paper, and thus I will omit them. Proposition 4 was conjectured by a Maple program that I wrote by modifying a program written by Professor Zeilberger.

Proposition 4. *The initial $6K-2$ terms of the sequence $\{a_{2K,1,K}(n)\}=\{a(n)\}$ given by the quadratic recurrence relation (1), i.e.:*

$$a(n)a(n-2K)=a(n-1)a(n-2K+1)+2a(n-K) \qquad (6)$$

with the initial conditions $a(n)=1$ for $n\le 2K$ are also generated by the following polynomials:

$$\begin{aligned}
a(i) &= 1 && \text{for } 1\le i\le 2K\\
a(2K+i) &= 2i+1 && \text{for } 1\le i\le K\\
a(3K+i) &= 1+2K+4i+2i^2 && \text{for } 1\le i\le K-1\\
a(4K+i) &= 4K^2i+2i^2+8Ki+6K^2+2i+10K-1 && \text{for } 0\le i\le K-1\\
a(5K+i) &= 12Ki^2+16K^2i+4K^2+4K^2i^2-2i^2+ &&\\
&\quad +48Ki+42K^2-10i+36K-9 && \text{for } 0\le i\le K-2
\end{aligned}$$

Proposition 5. *The sequence* $\{a_{2K,1,K}(n)\} = \{a(n)\}$ *defined by the quadratic recurrence relation (6), with initial conditions* $a(n) = 1$ *for* $n \leq 2K$ *is also generated by the linear recurrence relation defined by*

$$d(n) = \left[2K^2 + 6K - 1\right](d(n-2K+1) - d(n-4K+2)) + d(n-6K+3) \quad (7)$$

with the first $6K - 2$ *terms of* $\{d(n)\}$ *taken to be the first* $6K - 2$ *terms of the sequence* $\{a(n)\}$.

After proving the above, I experimented to see if the analogs to Propositions **??** and 3 seemed to be true for the k even case. It was indeed, and the following are those analogous Propositions.

Proposition 6. *Consider the sequence* $\{a_{2Ki,i,Ki}(n)\}_{n\in\mathbb{N}}$ *from (1), i.e.-* $\{a_{2Ki,i,Ki}(n)\} = \{a(n)\}$ *is defined by:*

$$a(n)a(n-2Ki) = a(n-i)a(n-(2K-1)i) + 2a(n-Ki) \quad (8)$$

with initial conditions $a(n) = 1$ *for* $n \leq 2Ki$. *Consider also the sequence* $\{a_{2K,1,K}(n)\} = \{d(n)\}$ *defined by*

$$d(n)d(n-2K) = d(n-1)d(n-(2K-1)) + 2d(n-K) \quad (9)$$

with initial conditions $d(n) = 1$ *for* $n \leq 2K$. *Then we have the following relationship*

$$a(Li+m) = d(L+1) \text{ for } m \in [1,i] \quad (10)$$

Proposition 7. *If the sequence* $\{a_{2Ki,i,Ki}(n)\} = \{a(n)\}_{n\in\mathbb{N}}$ *is given by equation (8) then it is also annihilated by the linear recurrence relation defined by*

$$\begin{aligned} a(n) &= a(n-1) + \sum_{l_1=1}^{2K-2}\Big(-a(n-l_1 i) + a(n-(l_1 i+1))\Big) + \\ &+ \sum_{l_2=2K-1}^{4K-3}\left[2K^2+6K-2\right]\Big(a(n-l_2 i) - a(n-(l_2 i+1)\Big) + \\ &+ \sum_{l_3=4K-2}^{6K-4}\Big(-a(n-l_3 i) + a(n-(l_3 i+1))\Big) \end{aligned}$$

Proofs of Propositions 6 and 7 are practically the same as the proofs of Propositions 2 and 3 respectively, thus they will be excluded.

3 Future Plans

I would like to eventually have a full characterization of sequences generated by recurrence relation (1). I have collected a lot of information from my program to find a minimal recurrence. I have many conjectured linear recurrence relations and would like to fit their coefficients with a function. Unfortunately I don't currently believe that a polynomial will do the job. I hope to research other types of functions, similar to polynomials, that may serve as the general coefficient for the linear recurrences.