INTRODUCTION TO MACAULAY 2

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The main source of information about Macaulay 2 is the webpage:
`http://www.math.uiuc.edu/Macaulay2/Manual/`

Macaulay 2 is installed on the machine `compute.rutgers.edu`, so you can ssh to there to use it. On the Macaulay 2 webpage are files you can download to install it on your own machine. For Macs, I've used the version at the Fink. For Windows machines, follow the directions at `commalg.org`.

To start Macaulay 2 on a Unix system, type `M2` at the command line.

You should see something like:

```
Macaulay 2, version 0.8.99
--Copyright 1993-2001, all rights reserved, D. R. Grayson and M. E. Stillman
--Factory 1.2c from Singular, copyright 1993-1997, G.-M. Greuel, R. Stobbe
--Factorization and characteristic sets 0.3.1, copyright 1996, M. Messollen
--GC 6.0 alpha 2, copyright, H-J. Boehm, A. Demers
--GNU C Library (glibc-2.2.1), copyright, Free Software Foundation
--GNU MP Library (gmp-3.1.1), copyright, Free Software Foundation

i1 :
```

The `i1 :` is the prompt.

The first thing we want to do is define a polynomial ring. Your main choice of fields is $\mathbb{Q}$ (written `QQ`) or a finite field $\mathbb{F}_p$ (written `ZZ/p`). The integers are represented by `ZZ`. To define a polynomial ring over $\mathbb{Q}$, we type:

```
i1 : R=QQ[a,b,c,d]
```

which produces:

```
o1 = R

o1 : PolynomialRing
```

The ring R is now the polynomial ring in three variables with coefficients in $\mathbb{Q}$. The default term order is graded reverse lexicographic. If you wanted to use the lexicographic order, for example, you would have instead typed:

```
i2 : S=QQ[a,b,c,d,MonomialOrder=>Lex]
```

To switch back to R type `use R`. To define an ideal in R type, for example $I = \langle c^2 - bd, bc - ad, b^2 - ac\rangle$, type:

```
i3 : I=ideal(c^2-b*d, b*c-a*d, b^2-a*c)

               2                          2
o3 = ideal (c  - b*d, b*c - a*d, b  - a*c)

o3 : Ideal of R
```

Notice the way the exponents are on a separate line in the output. This can be annoying - one way to avoid it is to type:

```
o4 : toString I

o4 = ideal(c^2-b*d,b*c-a*d,b^2-a*c)
```

This is my peculiarity - most Macaulay 2 users get around this problem by using it inside `emacs`. See the webpage for details.

To compute a Gröbner basis of the ideal type:

```
i5 : gb I

o5 = | c2-bd bc-ad b2-ac |

o5 : GroebnerBasis
```

To get the lead term ideal type:

```
i6 : leadTerm I

o6 = | c2 bc b2 |

               1       3
o6 : Matrix R  <--- R
```

The command `leadTerm` also finds the lead term of a polynomial.

To find the remainder when dividing a polynomial by a Gröbner basis type:

```
i7 : f=b*d-c^2

            2
o7 = - c  + b*d

o7 : R

i8 : f % I
```

```
o8 = 0

o8 : R

i9: g=b^3*c^3

       3 3
o9 = b c

o9 : R

i10 : g % I

       3 3
o10 = a d

o10 : R
```

The remainder on division (%) by I is the remainder on division by the *reduced* Gröbner basis for I with respect to the given term order.

You should now look at the documentation on the Macaulay 2 webpage, particularly the section labelled *Getting started*.

Exercises

(1) Compute a Gröbner basis for the ideal $\langle x^3y^2 - 4x^2y^3 + 5y^5, x^6 - 7xy^5 \rangle \subseteq \mathbb{Q}[x, y, z]$. Is $xy^9 \in I$?

(2) Explain how computing a Lex Gröbner basis for an ideal $I \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ can help you compute $I \cap \mathbb{Q}[x_i, x_{i+1}, \ldots, x_n]$.

(3) Let $I = \langle xy^2, y^3z \rangle$, and $J = \langle x^5, xy, z^4 \rangle$. Compute $I \cap J$ using the algorithm $I \cap J = (tI + (1-t)J) \cap \mathbb{Q}[x, y, z]$ (first prove this!). Check your answer using the command `intersect(I,J)`.

(4) This exercise explains how we can use Gröbner bases to solve polynomial equations.

   (a) Let $S = \mathbb{C}[x_1, \ldots, x_n]$ have the lexicographic term order. Let I be an ideal in S. Show that if I contains any polynomials containing only powers of x_n, then there must be one in the reduced Gröbner basis for I.

   (b) Let I be such that $V(I)$ is a finite set. Show that $I(V(I)) = \{f \in S : f(a) = 0 \text{ for all } a \in V(I)\}$ must contain a polynomial only containing only powers of x_n.

(c) The radical of I is $\{f \in S : f^n \in I \text{ for some } n > 0\}$. When we work over the complex numbers $I(V(I))$ is the radical of I (this is the Hilbert Nullstellensatz). Assuming this, show that I contains a polynomial containing only powers of x_n.

(d) If we know a polynomial in I containing only powers of one variable, we can solve for the roots of this polynomial (symbolically or numerically), and use this to reduce to a simpler problem. Use this idea to solve the system of equations:

$$\begin{aligned} x^2 - 3xy + y^2 &= 0 \\ x^3 - 8x + 3y &= 0 \\ x^2y - 3x + y &= 0 \end{aligned}$$

Give your answer symbolically (that is, in terms of radicals).

(5) Does the magic square property still hold if you remove the diagonal condition on a magic square? If not, how could you find a counterexample?

(6) Gröbner bases depend on the term order used. Compute the Gröbner basis for $I = \langle x^5 + y^4 + z^3 - 1, x^3 + y^2 + z^2 - 1\rangle$ with respect to the revlex and lexicographic term orders. Try the same thing for the ideal $J = \langle x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1\rangle$