Solutions to Dr. Z.'s Math 403 REAL Quiz 3

1. Find the values of the given expressions

a) (2 points) $e^{i\frac{3\pi}{4}}$

(b) (2 points) $Log(e^{10} - e^{10}i)$.

Sol. to 1(a): Using $e^{it} = \cos(t) + i\sin t$, we have

$$e^{i\frac{3\pi}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$
$$= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \quad .$$

Ans. to 1(a): $-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$.

Sol. to 1(b): $e^{10} - e^{10}i = e^{10}\sqrt{2}e^{i(-\pi/4)}$. Using

$$Log z = \log|z| + i\, Arg\, z \quad ,$$

since $|z| = e^{10}\sqrt{2}$ and $Arg\, z = -\pi/4$ (recall that $Arg\, z$ must be in $[-\pi, \pi)$), we have

$$Log(e^{10} - e^{10}i) = \log(e^{10}2^{1/2}) - i\frac{\pi}{4} = (10 + \frac{\log 2}{2}) - i\frac{\pi}{4} \quad .$$

Ans. to 1(b): $(10 + \frac{\log 2}{2}) - i\frac{\pi}{4}$.

2. (4 points) Show that $F(z) = e^z$ maps the strip

$$S = \{x + iy \,:\, -\infty < x < \infty\,,\, 0 \leq y \leq \pi/4\}$$

onto the region

$$\Omega = \{s + it \,:\, s \geq t \geq 0\,\}\backslash\{0\} \quad ,$$

and that F is one-to-one on S.

Sol. to 2: for $z \in \mathcal{S}$ $F(z) = e^{x+iy} = e^x(\cos y + i\sin y)$.

Of course it can never be 0 but the range are all the complex numbers whose absolute value is positive and whose argument is between 0 and $\pi/4$. That's exactly the region Ω. It is one-to-one, since if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, with $F(z_1) = F(z_2)$ then $e^{z_1 - z_2} = 1$ and hence $z_1 - z_2 = 2\pi inn$, with n integer. i.e $x_1 = x_2$ and $y_1 - y_2 = 2\pi n$. But this can never happen since $0 \leq y_1 \leq \pi/4$ and $0 \leq y_1 \leq \pi/4$, $y_1 - y_2$ is between $-\pi/4$ and $\pi/4$.