Solutions to First Practice Exam for Math 403(02) (Spring 2020, Rutgers University, Dr. Z.)

Version of May 3, 2020 (correcting a sign error in #12, thanks to Mike Murr Previous Version of May 2, 2020 (correcting a computational error (due to my doing "mental math" rather than using paper-and-pencil in the answer to #4). THANKS TO DHANVIN PATEL.

Every question is worth 15 points except the last one that is worth 20 points

1. Suppose that you have the equation

$$\frac{1}{Z^2} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{Z_5}$$

If

$$Z_1 = 2i \quad , \quad Z_2 = 3i \quad , \quad Z_3 = -2i \quad , \quad Z_4 = -3i \quad , \quad Z_5 = i$$

Find Z.

Sol. to 1. By plugging in the values and simplifying you get $1/Z^2 = -i$. So $Z^2 = i = e^{i\pi/2}$, $e^{5i\pi/2}$ so $Z = i = e^{i\pi/4}$, $e^{5i\pi/4}$ So $Z = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$, $Z = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}$

Ans. to 1: $\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$.

2. Find the contour integral of

$$f(z) = \int_\Gamma \frac{z^9}{(z-1)(z-i)}\,dz$$

If (a) Γ is the contour $|z-1-i| = \frac{1}{5}$

(b) Γ is the contour $|z| = 5$

(c) Γ is the contour $|z-i| = \frac{1}{2}$

(d) Γ is the contour $|z-1| = \frac{1}{2}$

Sol. to 2. The only poles are $z = 1$ and $z = i$. The residue theorem says that the contour integral over a closed curve Γ is $2\pi i$ times the sum of the residues of the poles INSIDE Γ.

(a): Neither poles are inside $|z-1-i| = 1/5$ so the integral is 0 (also follows from Cauchy's theorem)

(b): Both poles are in $\{z : |z| < 5\}$. The residue at $z = 1$ is $1^9/(1-i)$=$1/(1-i) = \frac{1}{2}(1+i)$. The residue at $z = i$ is $i^9/(i-1) = -i/(1-i) = -i(1+i)/2 = \frac{1}{2}(1-i))$ whose sum is 1, hence the integral equals $2\pi i$.

(c): The only pole in $|z-i| < \frac{1}{2}$ is $z = i$, the residue is $i^9/(i-1) = -i/(1-i) = -i(1+i)/|1+i|^2 = \frac{1}{2}(1-i)$. Hence the contour integral is $2\pi i(\frac{1}{2}(1-i)) = \pi + i\pi$

(d): The only pole in $|z-1| < 1/2$ is $z = 1$, the residue is $1^9/(1-i) = \frac{1}{2}(1+i)$ Hence the contour integral is $2\pi i \frac{1}{2}(1+i) = -\pi + i\pi$.
Ans. to 2) a) 0 b) $2\pi i$ c) $\pi + i\pi$ d) $-\pi + i\pi$.

3. By using the real form of Green's theorem

$$\int_\Gamma \{udx + vdy\} = \int\int_\Omega (v_x - u_y)\, dx\, dy \quad ,$$

(with the appropriate conditions)

Compute the (real) line integral

$$\int_\Gamma (3x^6 \, + \, 3y)\, dx \, + \, (11x \, + \, 7y^3)\, dy \quad ,$$

where Γ is the contour, transversed **clockwise** that is the boundary of the region in the xy-plane

$$\{ \, (x,y) \, : \, 100 \le x \le 120 \quad , \quad 1000 \le y \le 1003 \, \} \quad .$$

Sol. to 3. We have $u = 3x^6 + 3\,y$, $v = 11, x + 7\,y^2$. $v_x = 11, u_y = 3$, hence the integrand is $v_x - u_y = 8$. The region is a rectangle of area $(120-100)\cdot(1003-1000) = 60$. Since the area integral of a constant over any region is the area of that region times that constant, the area integral, and hence the desired line integral when the direction is the default direction (counterclockwise) is $8 \cdot 60 = 480$. But we are told that the direction of motion is CLOCKWISE, so we have to multiply by -1.

Ans. to 3: -480 .

4. By using the Taylor polynomial of degree 3 of $f(z) = \sin z$, find an approximation to $\sin \frac{i}{2}$.

Sol. to 4: The Taylor polynomial of degree 3 of $sin(z)$ is $z - z^3/6$

Plugging-in $z = \frac{i}{2}$ gives
$\frac{i}{2} - (1/6)\,(\frac{i}{2})^3 = \frac{i}{2} + \frac{i}{48} = \frac{25}{48}\, i$

Ans. to 4: $\frac{25}{48}\, i$.

5. Find the Taylor polynomial of degree 2 around $z_0 = i$ of $f(z) = z^4$

Sol. to 5: The Taylor polynomial of degree 2 of $f(z)$ at $z = z_0$ is

$$f(z_0) + f'(z_0)\,(z - z_0) + \frac{f''(z_0)}{2}\,(z - z_0)^2 \quad .$$

Here $z_0 = i$ and we have

$f(z) = z^4, f'(z) = 4\,z^3, f''(z) = 12\,z^2.$

$f(i) = i^4 = 1,\ f'(i) = 4 \cdot i^3 = -4i\ ,\ f''(i) = 12 \cdot i^2 = -12.$

Hence The Taylor polynomial of degree 2 of $f(z) = z^4$ at $z = i$ is

$1 - 4i(z - i) - 6(z - i)^2$

Ans. to 5: $1 - 4i(z - i) - 6(z - i)^2$.

6. Find the change of argument of

$$f(z) = \frac{e^z(z - i)(z - 2i)(z - 3i)}{(z - 10i)} \quad ,$$

(a) as z goes along, counterclockwise, the upper semi-disc $\{z : |z| = 5 \quad , \quad Im\, z \geq 0\}$

(b) as z goes along, **clockwise**, the upper semi-disc $\{z : |z| = 20 \quad , \quad Im\, z \geq 0\}$

Sol. to 6: Recall that the change of argument traveling COUNTER-CLOCKWISE along a closed contour of a function with at worst poles is the number of Zeros MINUS the number of poles INSIDE the contour. This is only valid if none of the zeros and none of the poles lie ON the contour.

Note that $f(z)$ has zeros at $z = i$, $z = 2i$, $z = 3i$ and a pole at $z = 10i$.

Sol. to a): None of the zeros and none of the poles lie on the contour. All three zeros lie inside the semi-circle, but the pole does not. Hence the required difference is $3 - 0 = 3$. Multiplying by 2π, we get that the desired change of argument 6π.

Sol. to b): None of the zeros and none of the poles lie on the contour. All three zeros lie inside the semi-circle, and so does the pole $z = 10i$. Hence the required difference is $3 - 1 = 2$. Multiplying by 2π, we get that the desired change of argument 4π . BUT, now we are told that the trip goes clockwise, so we have to multiply by minus one, getting -4π.

Ans. to 6: a) 6π b) -4π.

7. which of the following functions $f(z) = f(x + iy)$ are analytic. Explain

$$(a) \quad f(x+iy) = 2x - 3iy \quad , \quad (b) \quad f(x+iy) = -ix + y \quad ,$$

$$(c) \quad f(x+iy) = x^2 - y^2 + i(2xy) \quad , \quad (d) \quad f(x+iy) = x^2 + y^2 + i(2xy) \quad ,$$

Sol. to 7: We use the Cauchy-Riemman equations: $u_x = v_y, u_y = -v_x$.
For a): $u = 2x, v = -3y$, so $u_x = 2, v_y = -3$, these are NOT equal, so it is NOT analytic

For b): $u = y, v = -x$, so $u_x = 0, v_y = 0, 0 = 0$, OK!, also $u_y = 1, v_x = -1$ $1 = -(-1)$ also OK! Analytic

For c): $u = x^2 - y^2, v = 2\,xy$, so $u_x = 2x, v_y = 2x, 2x = 2x$, OK!, also $u_y = -2y, v_x = 2y$, $-2y = -(2y)$ also OK! Analytic

For d): $u = x^2 + y^2 2$, $v = 2xy$ so $u_x = 2x$, $v_y = 2x$, so far so good. Also $u_y = 2y, v_x = 2y$, So it is NOT true that $u_y = -v_x$ so this function flunks the second Cauchy-Riemann equation, so Analytic

Ans. to 7: The following ones are: (put the subset of $\{b, c\}$):

8. Use Rouché's theorem to prove that all the zeros of the function $f(z) = z^5 - i$ lie in the annulus $0.99 < |z| < 1.01$. Explain!

Sol. to 8: We compare $f(z) = z^5 - i$ to $h(z) = -i$ on —z—=0.99 $|z^5 - i + i| = |z^5| = |z|^5 = (0.99)^5 < |i|$ so the number of zeros of $f(z)$ in $|z| < 0.99$ is the same as the number of zeros of $h(z) = -i$, namely NO zeros.

We compare $f(z) = z^5 - i$ to $h(z) = z^5$ on $|z| = 1.01$. $|z^5 - i - z^5| = |-i| = 1 < 1.01^5 = |-z^5|$ on $|z| = 1.01$.

So the number of zeros of $f(z)$ in $|z| < 1.01$ is the same as the number of zeros of $h(z) = z^5$ in $|z| < 1.01$, namely 5 (counting multiplicity, of course).

By the Fundamental Theorem of Algebra (or for that matter, in this simple case, directly) $f(z)$ has altogether five zeros. None of them in $|z| < 0.99$, hence all of them are in the annulus $0.99 < |z| < 1.01$.

Comment: We can replace 0.99 by any positive real number less than 1 and replace 1.01 by any positive real number large than 1, no matter how close, so the above reasoning implies that the five zeros of $f(z) = z^5 - i$ all lie ON $|z| = 1$. Of course, this can be done directly w/o Rouche, just solve $z^5 = i$, like we did in Lecture 2, but the point of this problem is to test your knowledge of Rouche's theorem, and that you can apply it correctly.

9 Which of the following statements are **obviously wrong**, without actually doing the problem. EXPLAIN!

a) The change of argument, of the function $f(z) = e^{z^3}$ as it transverses, counter-clockwise, the circle $|z| = 10$ is 2π.

b) The change of argument, of the function $f(z) = \cos z$ as it transverses, counter-clockwise, the circle $|z| = 100$ is 3π.

c) The change of argument, of the function $f(z) = \frac{1}{z^9+1}$ as it transverses, **clockwise**, the circle $|z| = 5$ is 18π.

d) The change of argument, of the function $f(z) = \tan z$ as it transverses, counter-clockwise, the circle $|z| = 1$ is 2π.

a) This function has neither zeros nor poles, so the change of argument over ANY closed contour is ALWAYS ZERO. WRONG!

b) The change of argument is always an INTEGER (either positive or negative) times 2π. $\frac{3\pi}{2\pi} = \frac{3}{2}$ is **not** an integer, so it is WRONG!

c) $f(z) = 1/(z^9 + 1)$ has NO zeros, but 9 poles INSIDE $|z| = 5$, hence the change of argument as it travels the usual (default orientation is $2\pi(0 - 9) = -18\pi$. But here we are traveling CLOCKWISE, to the desired answer is 18π, so it is CORRECT.

d) $f(z) = \tan z = \frac{\sin z}{\cos z}$ has ONE zero in $|z| < 1$, namely $z = 0$ and NO poles there (the closest pole is $z = \frac{\pi}{2} = 1.57\ldots$)

So the change of argument is $2\pi(1 - 0) = 2\pi$. CORRECT.

Ans. to 9: The following are **wrong**: $\{a, b\}$.

10. A certain entire function maps the region $\{z : |z| < 1\}$ **onto** the region $\{z : |z - 3| \leq 2\}$. What can you say about such a function. EXPLAIN!

Sol. to 10: Such a function DOES NOT EXIST. If the domain is OPEN, then the range under an analytic function on that domain must be also OPEN. $z : |z| < 1$ is OPEN while $z : |z - 3| \leq 2$ is CLOSED, so NO WAY!

11. Evaluate the following integral

$$\int_0^{2\pi} (2i + 10\, e^{it})^9 \, dt$$

Sol. to 11: This is a special case of the MEAN VALUE THEOREM with $z_=2i$, $f(z) = z^9$ and $r = 10$. $f(z)$ is ALWAYS analytic, so the value of the integral equals
$2\pi f(2i) = 2\pi\,(2i)^9 = 2^10\pi\, i = 1024\,\pi\, i$

Ans. to 11: $1024\,\pi\,i$.

12: You are told that a certain function, analytic in $|z| < 10$ is such that $f(0) = 0$, $|f(z)| \leq 100$ and $f(5i) = (30 + 40i)$. Find the exact expression for $f(z)$.

Sol. to 12: Consider $g(z) = f(10z)/100$. $g(z)$ is analytic in $|z| < 1$, $g(0) = 0$ and $|g(z)| <= 1$.

By Schwarz's Lemma we have the stronger inequality $|g(z)| <= |z|$. By the **second part** of Schwarz, if it so happens that there exists z_0 such that $|z_0| < 1$ and $|g(z_0)| = |z_0|$ then $g(z) = cz$ for some constant c (with, of course $|c| = 1$).

Here $|g(\frac{i}{2})| = |30 + 40i|/100 = 1/2$, so

$g(z) = cz$. We still need to find c.

$g(\frac{i}{2}) = \frac{3+4i}{10}$.
So $c = \frac{4}{5} - i\frac{3}{5}$ and
$g(z) = (\frac{4}{5} - i\frac{3}{5})z$, hence
$f(z) = 100\,g(z/10) = 100(\frac{4}{5} - i\frac{3}{5})(z/10) = (8 - 6i)\,z$

Ans. to 12: $f(z)$ must be $f(z) = (8 - 6i)\,z$.

COMMENT: A somewhat quicker way to solve this problem is not change to g(z). Because of Rouchẃe know that $f(z) = cz$ for SOME constant c, and one can directly use the fact that

$f(5i) = 30 + 40i$ to find that constant.

13: Find the exact value of the contour integral

$$\int_{|z-2i|=5} \frac{e^{z^2}}{(z-i)^2}\,dz$$

Sol. to 13:For any analytic function the integral of $f(z)/(z - z_0)^(k + 1)$ over any contour Γ containing z_0 is $f^{(k)}(z_0)/k!$ TIMES $2\pi\,i$. Here

$f(z) = e^{z^2}$, $z_0 = i$, $k = 1$.
By the chain rule:
$f'(z) \;=\; 2z\,e^{z^2}$,

so $f'(i) = 2ie^(i^2) = 2i\,e^{-1}$.

Multiplying by $2\pi i$, we get

$(2ie^{-1})(2\pi i) = -4\frac{\pi}{e}$.

Ans. to 13: $-4\frac{\pi}{e}$**.**