Solutions to Dr. Z.'s Intro to Complex Variable Attendance Quiz for Lecture 6

1. Show that $F(z) = e^z$ maps the strip

$$\mathcal{S} = \{z = x + iy \, : \, -\infty < x < \infty \, , \, -\pi/2 \leq y \leq \pi/2\} \quad ,$$

onto the region

$$\Omega = \{w = u + iv \, : \, u \geq 0 \, , \, w \neq 0\}$$

and that F is one-to-one on $\mathcal{S}$.

Furthermore, show that F maps the boundary of $\mathcal{S}$ onto all the boundary of Ω except $w = 0$. Explain what happens to each of the horizontal lines $\{z \, : \, Im\, z = \pi/2\}$ and $\{z \, : \, Im\, z = -\pi/2\}$.

Sol. to 1: Writing $z = x + iy$, $-\infty < x < \infty$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, we have

$$w = u + iv = e^z = e^{x+iy} = e^x(\cos y + i \sin y) = e^x \cos y \, + \, ie^x \sin y \quad .$$

Since $\cos y$ is never negative in $-\pi/2 \leq y \leq \pi/2$, the range of F is the right half-plane, $Re\, w \geq 0$, **except** $w = 0$, since e^z is never zero.

It is **onto** since if $w = u + iv$, with $u \geq 0$, we can take $x = \log(|w|)$, and there is a always a y between $-\pi/2$ and $\pi/2$ such $\tan y = u/v$. It is **one-to-one**, since whenever $e^{z_1} = e^{z_2}$ we have $Re\, z_1 = Re\, z_2$ and $Im\, z_1 - Im\, z_2$ is an integer multiple of 2π. Since the width of the strip is π we are safe, and for every w in the range there is only **one** z such that $w = F(z)$.

Regarding the boundary, when $Im\, z = y = \frac{\pi}{2}$, we have

$$F(z) = e^{x+i\frac{\pi}{2}} = e^x \cos(\frac{\pi}{2}) + ie^x \sin(\frac{\pi}{2}) = e^x \cdot 0 + ie^x \cdot 1 = ie^x \quad ,$$

and since $e^x > 0$ we see that the horizontal line $y = \frac{\pi}{2}$ gets mapped to the open half-line $v > 0$.

When $Im\, z = y = -\frac{\pi}{2}$, we have

$$F(z) = e^{x-i\frac{\pi}{2}} = e^x \cos(-\frac{\pi}{2}) + ie^x \sin(-\frac{\pi}{2}) = e^x \cdot 0 + ie^x \cdot (-1) = -ie^x \quad ,$$

and since $e^x > 0$ we see that the horizontal line $y = -\frac{\pi}{2}$ gets mapped to the open half-line $v < 0$.