Solutions to Dr. Z.'s Intro to Complex Variable Attendance Quiz for Lecture 21

1. State and Prove the Maximum Modulus Principle.

(Hint: you can use the intuitively obvious fact that if $f(z)$ is a non-constant analytic function on a domain D, then the range of $f(D)$ is an open set).

Sol. to 1: The **Maximum Modulus Principle** says that if f is a **nonconstant** analytic function on a domain D, then $|f|$ can have no local maximum on D.

In other words the maxima of $|f|$ on the *clusure* of D must be on the boundary of D.

Proof: Suppose, on the contrary, that there is a local maximum $z = z_0$ **inside** D. Since D is **open**, it means that there is some positive r (possibly small) such that the disc $|z - z_0| < r$ is contained in D and $|f(z_0)| \geq |f(z)|$ for the points z inside that disc.

But this means that $f(z_0)$ lies on the **boundary** of the **image** of $|z - z_0| < r$ under the function $f(z)$, namely the open set

$$W = \{f(z) \, : \, |z - z_0| < r\} \quad ,$$

But W is an open set containing $f(z_0)$, contradiction!

2. Let $f(z) = 2z^3/(z + 2)^4$; find the maximum value of $|f(z)|$ as z varies over the disc $|z| \leq 1$.

Sol. to 2: By the **maximum principle**, the maximum **value** of $|f(z)|$ must lie on the **boundary** of the disc $|z| \leq 1$, in other words on the **circle** $|z| = 1$.

On that disc

$$|f(z)| = \frac{2|z|^3}{|z+2|^4} = \frac{2}{|z+2|^4} \quad .$$

Recall that $|z+2| = |z-(-2)|$ is the **distance** of z from the point -2 (alias $(-2,0)$). The **shortest** distance is acheived at the point $z = -1$ (alias $(-1, 0)$), and that distance is 1 (draw a picture!). Hence the minimum of $|z + 2|$ as z goes over $|z| = 1$ is 1, hence the **maximum** of the reciprocal $1/|z + 2|$ is 1 and the maximum of $1/|z + 2|^4$ is 1. Plugging it in, we have:

Ans. to 2.: The maximum value of $| = 2z^3/(z + 2)^4|$ as z varies over the disc $|z| \leq 1$. is 2.