Solutions to Dr. Z.'s Intro to Complex Variable Attendance Quiz for Lecture 19

1. Use the argument principle to determine the number of zeros of f in the first quadrant of the following function:

$$f(z) = z^5 + 5z + 11 \quad .$$

Sol. 1: Consider the contour consisting of the circumference of the quarter circle

$$\{(x, y) \, : \, x^2 + y^2 = R^2 \, , \, x \geq 0 \, , \, y \geq 0\}$$

. Assume that R is **very** large . Of course, as usual, we travel it, starting at the origin, in the **counter-clockwise** direction.

It consists of three portions.

• The portion on the positive real axis from $(0, 0)$ to $(R, 0)$.

Here $f(x) = x^5 + 5x + 11$ is always positive, and the argument is always 0, so there is no change.

• The circular arc $z = Re^{it}$, with $0 < t < \pi/2$. When R is very big, f is dominated by z^5, so this equals $(Re^{it})^5 = R^5 \, e^{5\,i\,t}$. So as t goes from 0 to $\pi/2$, the argument changes from 0 to $5\pi/2$. So it goes full circle (2π) and an extra $\pi/2$ counter-clockwise.

• The line segment on the y axis from $(0, R)$ back to the origin $(0, 0)$. Here $z = iy$ and

$$f(z) = (iy)^5 + 5iy + 11 = 11 + i(y^5 + 5y) \quad ,$$

This is always in the first quadrant. When $y = R$ it is practically on the y axis (so the argument is $\pi/2$) but as y goes down, it slides down along the line $x = 11$, winding up at $(11, 0)$, so the argument changes from $\pi/2$ down to 0.

So the **total** change in the argument is $5\pi/2 - \pi/2 = 2\pi$. (Note that first the argument went up, from 0 to $5\pi/2$ and then it went down by $\pi/2$, to 2π).

By the **argument principle**, the total number of zeros inside this contour, when R is very big is $2\pi/(2\pi) = 1$, and hence that is the number of zeros in the first quarant.

Ans. to 1: The number of zeros of $f(z) = z^5 + 5z + 11$ in the first quadrant is one.