Solutions to Dr. Z.'s Intro to Complex Variable Attendance Quiz for Lecture 18

1. Compute the following integral

$$\int_0^{2\pi} \frac{dt}{1+\cos^2 t}$$

Sol. 1: We use the fact that $\cos t = (z+z^{-1})/2$ where $z=e^{it}$, so $dz = ie^{it}dt = izdt$, so $dt = \frac{dz}{iz}$.

The integral becomes the **contour integral** over the unit circle $|z|=1$.

$$\frac{1}{i}\int_{|z|=1} \frac{dz}{(1+(z+1/z)^2/4)z} = \frac{1}{i}\int_{|z|=1} \frac{4\,dz}{(4+(z+1/z)^2)z} = \frac{1}{i}\int_{|z|=1} \frac{4\,dz}{(4+z^2+2+z^{-2})z} =$$

$$\frac{1}{i}\int_{|z|=1} \frac{4\,dz}{(z^2+6+z^{-2})z} = \frac{1}{i}\int_{|z|=1} \frac{4z\,dz}{(z^4+6z^2+1)} \quad .$$

$\frac{4z\,dz}{(z^4+6z^2+1)}$ is a **rational function**. Let's find its poles. Putting $z^2=X$ we get the quadratic equation

$$X^2+6X+1=0 \quad ,$$

whose roots are

$$\frac{-6\pm\sqrt{6^2-4\cdot1\cdot1}}{2\cdot1} = \frac{-6\pm\sqrt{32}}{2} = -3\pm\sqrt{8} \quad .$$

So there are two roots $-3-\sqrt{8}$ whose absolute value is more than 1 and $-(3-\sqrt{8})$ whose absolute value is less than 1.

Going back to the roots of $z^4+6z^2+1=0$ the two roots that are inside the unit circle , and hence the poles of the integrand that are in $|z|<1$ are

$$z_1 = (\sqrt{3-\sqrt{8}})i \quad , \quad z_2 = -(\sqrt{3-\sqrt{8}})i \quad ,$$

(Note that $z_1^2 = z_2^2 = -(3-\sqrt{8})$).

We need the residue of $\frac{4z}{(z^4+6z^2+1)}$ at $z=z_1$ and $z=z_2$.

Recall that the residue of a rational function $P(z)/Q(z)$ at a simple pole $z=z_1$ is $P(z_1)/Q'(z_1)$. Here $P(z)=4z$ and $Q(z)=z^4+6z^2+1$ so $Q'(z)=4z^3+12z$ and it follows that the residue is $\frac{4z_1}{4z_1^3+12z_1} = \frac{1}{z_1^2+3}$

But $z_1^2=\sqrt{8}-3$ so $z^1+3=\sqrt{8}$ and the residue is $\frac{1}{\sqrt{8}}$. Similarly, the residue at $z=z_2$ is also $\frac{1}{\sqrt{8}}$.

It follows that our integral equals

$$(2\pi i)\frac{1}{i}\frac{2}{\sqrt{8}} = \frac{4\pi}{\sqrt{8}} = \sqrt{2}\pi \quad .$$

Ans. to 1: $\int_0^{2\pi} \frac{dt}{1+\cos^2 t} = \sqrt{2}\pi$.