Solutions to Dr. Z.'s Intro to Complex Variable Attendance Quiz for Lecture 16

1. Find the Laurent expansion, in powers of z and $\frac{1}{z}$ of the following rational function

$$f(z) = \frac{1}{(z+1)(z+2)} \quad ,$$

a) in $1 < |z| < 2$ **b)** in $2 < |z| < \infty$.

Sol. of 1: We first, rewrite $f(z)$ using **partial fractions**, like we did way back in calc2.

$$\frac{1}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2} \quad .$$

This means

$$\frac{1}{(z+1)(z+2)} = \frac{A(z+2)+B(z+1)}{(z+1)(z+2)} = \frac{(A+B)z+(2A+B))}{(z+1)(z+2)} \quad ,$$

Comparing numerators

$$1 = (A+B)z + 2A + B \quad .$$

Hence $A + B = 0$, $2A + B = 1$, hence $B = -A$, hence $A = 1$ and $B = -1$. We get

$$\frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2} \quad .$$

The only pole of $\frac{1}{z+2}$ is at $z = -2$ so it is analytic in $|z| < 2$, and we can express it as a Taylor series in z about $z = 0$.

$$\frac{1}{z+2} = \frac{1}{2(1+z/2)} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(\frac{z}{2})^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}z^n \quad ,$$

valid in $|z| < 2$, and hence also in $1 < |z| < 2$.

On the other hand $1/(z+1)$ does not have a Taylor expansion in $1 < |z| < 2$, so we have to express it in powers of $1/z$. We write

$$\frac{1}{1+z} = \frac{1}{1+1/(1/z)} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} \quad .$$

valid for $1 < |z|$ and hence in $1 < |z| < 2$. Combing we have

$$\frac{1}{(z+1)(z+2)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}z^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} z^n \quad .$$

Ans. to 1a: The Laurent expansion of the rational function $\frac{1}{(z+1)(z+2)}$ in the region $1 < |z| < 2$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} z^n \quad .$$

Sol. to 1b: When $|z| > 2$ we need to express both $1/(z+1)$ and $1/(z+2)$ in powers of $1/z$. We already have done it for $1/(z+1)$. We now have to do it for $1/(z+2)$.

$$\frac{1}{2+z} = \frac{1/z}{(1+2/z)} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-2)^n}{z^n} = \sum_{n=0}^{\infty} \frac{(-2)^n}{z^{n+1}} \quad .$$

Combining, we have

$$\frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{(-2)^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n - (-2)^n}{z^{n+1}} \quad .$$

Ans. to 1b: The Laurent expansion of the rational function $\frac{1}{(z+1)(z+2)}$ in the region $2 < |z| < \infty$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n - (-2)^n}{z^{n+1}} \quad .$$

Note that it only has negative powers of z.