Solutions to Dr. Z.'s Intro to Complex Variable Attendance Quiz for Review Session for Exam 1

1. Consider the mapping $f(z) := z^6$ defined on the **domain**

$$D = \{z = re^{i\theta} \, : \, 0 \leq r < \infty \, , \, 0 \leq \theta \leq \frac{\pi}{6} \} \quad .$$

(a) Find the *image* (alias *range*) $f(D)$, and prove that it is indeed the image, i.e. that it is **onto**.

(b) Prove that the mapping is one-to-one.

Sol. of 1(a): Since $f(re^{i\theta}) = r^6 e^{i(6\theta)}$ and $0 \leq 6\theta \leq \pi$, it is reasonable to guess that the image is

$$f(D) = \{z = re^{i\theta} \, : \, 0 \leq r < \infty \, , \, 0 \leq \theta \leq \pi \} \quad ,$$

that happes to be the **upper-half plane**, $Im\, z \geq 0$.

To prove that it is onto, let $w = Re^{i\theta}$ (with $R \geq 0$ and $0 \leq \theta \leq \pi$). We have to show that there exists $z \in D$ such that $f(z) = w$. Putting $z = re^{i\alpha}$, we have

$$r^6 \, e^{6i\alpha} = Re^{i\theta} \quad .$$

Hence $r^6 = R$ and $6\alpha = \theta$ and we get $r = R^{1/6}$ and $\alpha = \theta/6$ (since $0 \leq \theta \leq \pi$, $0 \leq \alpha \leq \frac{\pi}{6}$), so $z \in D$.

Sol. to 1(b)

Suppose that

$$f(z_1) = f(z_2) \quad .$$

Write

$$z_1 = r_1 e^{i\theta_1} \quad , \quad z_2 = r_2 e^{i\theta_2} \quad ,$$

with $0 \leq r_1 < \infty$, and $0 \leq r_2 < \infty$ and $0 \leq \theta_1 \leq \frac{\pi}{6}$, $0 \leq \theta_2 \leq \frac{\pi}{6}$. Hence

$$r_1^6 e^{6i\theta_1} = r_2^6 e^{6i\theta_2} \quad .$$

Comparing the absolute values we get

$$r_1^6 = r_2^6 \quad ,$$

and since r_1 and r_2 are non-negative real numbers, we have $r_1 = r_2$.

Hence

$$e^{6i\theta_1} = e^{6i\theta_2} \quad .$$

Hence

$$e^{6i(\theta_1 - \theta_2)} = 1 \quad .$$

Hence

$$6(\theta_1 - \theta_2) = 2\pi n \quad ,$$

with n being an integer. But (assuming, without loss of generality that $\theta_1 \geq \theta_2$), since both of them are between 0 and $\frac{\pi}{6}$, we have $0 \leq \theta_1 - \theta_2 \leq \frac{\pi}{6}$ and hence $0 \leq 6(\theta_1 - \theta_2) \leq \pi$, so n must be 0, hence $\theta_1 = \theta_2$.

Since $r_1 = r_2$ and $\theta_1 = \theta_2$ it follows that $z_1 = z_2$, proving that it is one-to-one.