Solutions to Dr. Z.'s Intro to Complex Variable Attendance Quiz for Lecture 11

1. Prove the fact that $\int_\Gamma f(z)\,dz = 0$ for the special case where Γ is the circle $|z| = R$, and $f(z) = z^n$, $f(z) = z^n$ where n is a non-negative integer.

Sol. to 1: On the boundary of the circle $z = Re^{i\theta}$, $0 \leq \theta \leq 2\pi$. So $dz = Re^{i\theta} i d\theta$. We have

$$\int_\Gamma f(z)\,dz = \int_0^{2\pi} R^n e^{in\theta} iRe^{i\theta} id\theta = R^{n+1} i \int_0^{2\pi} e^{i(n+1)\theta} d\theta$$

$$= R^{n+1} i \frac{e^{i(n+1)\theta}}{i(n+1)} |_0^{2\pi} = R^{n+1} i \frac{e^{i(n+1)(2\pi)} - 1}{i(n+1)} = 0 \quad ,$$

since $n+1 > 0$ we do not divide by 0 and here we used Euler's famous result that $e^{2i\pi} = 1$ and hence $e^{2ik\pi} = 1$ for all integers k.

Comment: The above is true for any integer n, positive or negative, *except* for $n = -1$. Then it equals $2\pi i$.