Answers to Dr. Z.'s Introduction to Complex Variable

Disclaimer: Not responsible for any errors. One dollar award to the **first** to find any error.

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hw1

1. (a) $2i$; (b) $22 + 14i$; (c) -2 ; (d) 1 .

2. (a) $\frac{1}{10} - \frac{3}{10}i$; (b) $-\frac{13}{500} + \frac{9}{500}i$

3. (a) $\sqrt{2}(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}))$; (b) $5(\cos(\tan^{-1}\frac{4}{3}) + i\sin((\tan^{-1}\frac{4}{3})$;

4. (a) $\frac{x}{x^2+y^2}$; (b) $-\frac{y}{x^2+y^2}$.

hw2

2. (a) $1, -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\; -\frac{1}{2} - \frac{\sqrt{3}}{2}\,i$;

(b)

$$2\cos(\frac{\pi}{5}) + 2\sin(\frac{\pi}{5})\,i \quad , \quad 2\cos(\frac{3\pi}{5}) + 2\sin(\frac{3\pi}{5})\,i \quad , \quad 2\cos(\frac{5\pi}{5}) + 2\sin(\frac{5\pi}{5}) \quad ,$$
$$2\cos(\frac{7\pi}{5}) + 2\sin(\frac{7\pi}{5})\,i \quad , \quad 2\cos(\frac{9\pi}{5}) + 2\sin(\frac{9\pi}{5})\,i \quad .$$

(c) $2^{1/4}\cos(\frac{\pi}{8}) + 2^{1/4}\sin(\frac{\pi}{8})\,i, -2^{1/4}\cos(\frac{\pi}{8}) - 2^{1/4}\sin(\frac{\pi}{8})\,i$.

3 $\frac{7}{3}, \frac{2}{3}$.

hw3

1. (a) $\{z = x + iy : x > 1 \quad , \quad y < 2\}$, $\{z = x + iy : x = 1 \quad , \quad y < 2\} \cup \{z = x + iy : x > 1 \quad , \quad y = 2\}$. (b) closed (c) yes

2. (a) $\{z = x + iy : x > 1 \quad , \quad y < 2\}$, $\{z = x + iy : x = 1 \quad , \quad y < 2\} \cup \{z = x + iy : x > 1 \quad , \quad y = 2\}$. (b) neither (c) yes

3. (a) $\{z = x + iy : x < 1 \quad , \quad y < 2\}$, $\{z = x + iy : x = 1 \quad , \quad y < 2\} \cup \{z = x + iy : x < 1 \quad , \quad y = 2\}$. (b) yes (c) yes

4. (a) $\{z : |z - 2| < 3\}$, $\{z : |z - 2| = 3\}$ (b) closed (c) yes

5. (a) $\{z : |z - 2| > 3\}$, $\{z : |z - 2| = 3\}$ (b) closed (c) yes

6. (a) $\{z : |z-2| < 1 \quad or \quad |z-5| < 2\}$, $\{z : |z-2| = 1 \quad or \quad |z-5| = 2\}$ (b) closed (c) no

hw4

1. (a). 0 (b). diverges (z_n travel along the unit cicle $|z| = 1$) c. diverges to ∞.

2. (a). The whole complex plane $\mathbf{C}$ (b) everywhere except $z = 1$ (c) The whole complex plane $\mathbf{C}$

4. (a). Yes (use the ratio test) (b). diverges (the indvidual terms do not go to 0) (c) Yes (take absolute value and then use the p test from calc2).

hw5

1. (a) $\frac{1}{2} - i\frac{\sqrt{3}}{2}$; (b) $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$;

(c) $\log 2 + i(\frac{\pi}{3} + 2k\pi)$, $k = 0, \pm 1, \pm 2 \ldots$ (d) $\frac{5}{2}\log 2 - \frac{\pi}{4}\,i$.

hw6

These are all ‘showing.’

hw7

1. (a) 0 (b) $-\frac{2}{3}$ (c) $\sin(1+i)$ (d) $2i\pi$

2. 2π if $k = 0$; 0 if $k \neq 0$.

hw 8

1. (a) $-2\,\sin(2\,z)$ **(b)** $2\,\cos(2\,z)$ **(c)** $-\csc^2 z$ **(d)** $\sec z\,\tan z$ **(e)** $\frac{1}{1+z^2}$

3. (a) $x^2 - y^2$, $2xy$ **(b)** $e^{-y}\cos x$, $e^{-y}\sin x$ **(c)** $x^3 - 3xy^2$, $3x^2y - y^3$

hw 9

1. (a) $z = 1$,$R = 1$ **(b)** $z = 5$,$R = 27$ **(c)** $z = 1$,$R = 1$ **(d)** $z = 0$,$R = 1$.

2. (a) $\sum_{k=0}^{\infty} \frac{(-2)^k}{k!}\, z^{2k}$; **(b)** $\sum_{k=0}^{\infty} z^{5k+3}$; **(c)** $\sum_{k=0}^{\infty} \frac{1}{(2k)!}\, z^{6k}$; **(d)** $1 + \sum_{k=1}^{\infty} \frac{2^{2k-1}}{(2k)!}\, z^{2k}$.

hw 10

3 a. $1+z-\frac{1}{3}z^3-\frac{1}{6}z^4+\ldots$ **b.** $1+\frac{1}{2}z-\frac{1}{8}z^2-\frac{13}{48}z^3-\frac{79}{384}z^4+\ldots$ **c.** $1+0\cdot z++0\cdot z^2++0\cdot z^3++0\cdot^4+\ldots$

d. $0 + 0\cdot z + 0\cdot z^2 + 0\cdot z^3 + z^4 + \ldots$ **e.** $0\cdot +z - \frac{1}{2}z^2 + \frac{5}{6}z^3 - \frac{1}{2}z^4 + \ldots$

hw 11

4. $2\pi i$ if $k = 1$, 0 for $k \geq 2$.

hw 12

1 (a) 0 **(b)** $2\,\pi\,i$ **(c)** 0 **(d)** $2\,\pi\,i\,\cos\frac{1}{2}$ **(e)** $2\,\pi\,ie^{i}$ or (more explictly) $-2\pi\,\sin\,1 + (2\pi\cos\,1)\,i$
.

2 (a) $\frac{2\pi}{\sqrt{15}}$ **(b)** $\frac{2\pi}{\sqrt{7}}$ **(c)** $\frac{\sqrt{2}\,\pi}{2}$.

hw13

1 (a) $z = \frac{\pi}{2} + n\pi$, n integer; order=1 **(b)** $z = 2\pi\,i\,n$, n integer; order=3 **(c)** $z = 2$, order 4 ; $z = 3$, order 4 **(d)** $z = 0$, order 4 ; $z = \pi\,n$, n integer, $n \neq 0$, order 2 **(e)** $z = 0$, order 1

2 (a) $\sum_{n=1}^{\infty} \frac{2^n}{n!}\, z^{n+3}$ $|z| < \infty$ **(b)** $-2i - 4(z-i) + 3i(z-i)^2 + (z-i)^3$, $|z| < \infty$ **(c)** $\sum_{n=0}^{\infty} 2^n\, z^{n+3}$, $|z| < \frac{1}{2}$ **(d)** $1 + 0\cdot z + \frac{1}{2}z^2 + 0\cdot z^3 + \frac{5}{24}z^4 + \ldots$, $|z| < \frac{\pi}{2}$.

hw 14 N/A

15

1 (a) $z = 0$, removable **(b)** $z = 0$, essential **(c)** $z = 1$, pole of order 2; $z = 2$, pole of order 1; $z = 3$, pole of order 4 **(d)** $z = 1$, removable **(e)** $z = \pi\,(2n+1)$ (n integer), all poles of order 1 **(f)** $z = \pi\,n$ (n integer), all poles of order 1 .

2

(a)

$$\frac{1}{2}\cdot\frac{1}{z^2} + \frac{4}{3}\,\frac{1}{z} + \sum_{n=0}^{\infty}\frac{2^{n+4}}{(n+4)!}\,z^n \quad ,$$

residue=$\frac{4}{3}$.

(b)

$$\frac{1}{2}\,\frac{1}{z+2} + \sum_{n=0}^{\infty}\frac{-1}{2^{2n+3}}\,(z+2)^n \quad ,$$

residue=$\frac{1}{2}$.

(c)

$$\frac{1}{z} + \frac{1}{3}\,z + \frac{1}{15}\,z^3 + \frac{2}{189}\,z^5 + \ldots$$

residue=1 .

(d)

$$\frac{1}{2}\,\frac{1}{z^2} + \frac{1}{6} + \frac{1}{30}z^2 + \frac{1}{189}z^4 + \ldots$$

residue=0 .

16

4 (a)

$$\sum_{n=0}^{\infty} \frac{1}{n! z^{2n}} \quad .$$

(b)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{1}{z^{4n-6}} = z^6 - \frac{1}{2} z^2 + + \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} \frac{1}{z^{4n-6}}$$

(c) $\sum_{n=0}^{\infty} \frac{6 \cdot 9^n}{z^{n+2}}$.

5

(a) $\sum_{n=0}^{\infty} \frac{(-2)^n}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{3^{n+1}}$,

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n (2^n - 3^n)}{z^{n+1}}$.

6 ((**a**) $\frac{1}{6} \cos 1 + \frac{1}{6} \sin 1$ (**b**) $-\frac{1}{8}$ (**c**) $\frac{1}{24}$

17

1 (a) 0 **(b)** $-\frac{72}{\pi} i$ **(c)** $\frac{24(2\sqrt{2}-3)}{\pi} i$

2 (a) $\frac{\sqrt{2}\pi}{2}$ **(b)** $\frac{\pi}{\sqrt{2\sqrt{3}-2}}$ **(c)** $\frac{\pi}{30}$.

3 (a) $\frac{\pi}{e}$ **(b)** $-\frac{\pi \sin(1)}{e}$.

18

1 $\frac{2\sqrt{15}\pi}{15}$. **2** $\frac{\pi\sqrt{3}}{6}$. **3** $\frac{1}{3}(\frac{2\sqrt{15}\pi}{15} - \frac{\pi\sqrt{3}}{6})$

4 $\frac{\sqrt{30}\pi}{15}$. **5** $\frac{\sqrt{42}\pi}{21}$. **6** $\frac{\sqrt{30}\pi}{15} - \frac{\sqrt{42}\pi}{21}$.

19

1 a) one ; **b)** one **c)** zero . **d)** one .

2 a) four ; **b)** three **c)** four **d)** three .

20

1: one **2** four.

21

2: 2 **3:** $2e^{\frac{1}{2}} = 3.297445\ldots$ **4:** 18 **5:** 1 .

22

3 (Hint: consider $g(z) = f(6z)/1000$) **4 a):** 0 **b):** $2\pi(\cos 1 + i \sin 1)$ **c):** 2π . **5** max value: 1, at $(x, y) = (1, 0)$; min value: -4, at $(x, y) = (1, 1)$.