Dr. Z.'s Intro to Complex Variable Lecture 20 Notes: The Zeros of an Analytic Function (Part II)

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Recall from the last lecture the very important

Argument Principle (Theorem 2 on p. 176) Suppose that h is analytic in a domain D except at a finite number of poles. Let γ be a piecewise smooth positively oriented simple closed curve in D which does not pass through any pole or zero of h and whose inside lies in D. Then (**counting multiplicities**)

$$(Number \quad of \quad Zeros \quad inside \quad \gamma) - (Number \quad of \quad Poles \quad inside \quad \gamma)$$

equals the change of $arg\, h(z)$ as z travels along γ , divided by 2π.

In particular, if $h(z)$ is analytic (e.g. a polynomial), it has no poles, so to find the number of zeros (counting multiplicities) you just observe how the argument of $h(z)$ changes as z travels along the curve.

So far, all our examples were about finding the number of zeros of **polynomials** in quarter or half planes. The next example is about a **transcendental function**.

Problem 20.1: Use the argument principle to find the number of solutions of the equation

$$2z + e^{-3z} = 2 \quad ,$$

in the right half-plane $\{z \, : \, Re\, z > 0\}$.

Solution: We need to find the number of zeros of the function $f(z) = 2z + e^{-3z} - 2$ in the right half plane. We consider the left semi-circle $|z| = R$, where R is HUGE. (See Fig. 3.4 on p. 176 of Stephen Fisher's book for the diagram) .

We move around it, starting at $z = -iR$ (alias $(0, -R)$) we go around the circular arc $z = Re^{i\theta}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, ending at $z = iR$ (alias $(0, R)$), and then we go down the y-axis back to the starting point $z = -iR$.

In the circular arc, $f(z)$ is practically $2z$ since $|e^{-3z}| = |e^{-3x+3iy}| = e^{-3x} < 1$ since $x > 0$, and $|2z| = 2R$. So the argument is (practically) the same as $2Re^{i\theta}$, that is θ. So the change in argument along the circular arc is π.

Since

$$f(iy) = 2iy + e^{-3iy} - 2 = (\cos\, 3y - 2) + i(2y - 2) \quad ,$$

its real part is always negative (in fact less than -1) and the imaginary part starts out very positive (so it is very up in the second quadrant, and then it slides down going very low in the third quadrant). In other words, it increases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$. So the net gain is π.

Adding the two contributions, we get that the change of argument is $\pi + \pi = 2\pi$, and dividing by 2π we get that the number of zeros of $f(z) = 2z + e^{-3z} - 2$ which is the same as the number of roots of the equation $2z + e^{-3z} = 2$ is exactly one.

Ans. to 20.1: The number of solutions of the equation $2z + e^{-3z} = 2$ in the right half plane is one.

We now have a **very important theorem**. Its proof is simple enough that you should know how to reproduce it.

Rouché's Theorem Suppose that f and g are analytic on an open set containing a piecewise smooth simple closed curve γ and its inside. If

$$|f(z) + g(z)| < |f(z)| \quad for \quad all \quad z \in \gamma \quad ,$$

then f and g have an equal number of zeros inside γ, counting multiplicities.

The proof is very elegant. First observe that neither $f(z)$ nor $g(z)$ can have zeros on γ (If there exists $z_0 \in \gamma$ such that $f(z_0) = 0$ and $g(z_0) \neq 0$ then $|NonZero| < 0$ nonsense, if there exists $z_0 \in \gamma$ such that $f(z_0) \neq 0$ and $g(z_0) = 0$ then $|f(z_0)| < |f(z_0)|$ nonsense, if there exists $z_0 \in \gamma$ such that both $f(z_0) = 0$ and $g(z_0) = 0$ then $0 < 0$ nonsense).

Assume WLOG that all comon zeros of $f(z)$ and $g(z)$ have been canceled (if not cancel them out, it does not change the **difference** between their number of zeros).

Let $h = g/f$. Dividing the condition $|f(z) + g(z)| < |f(z)|$ by $|f(z)|$ we get

$$|h(z) + 1| < 1 \quad for \quad all \quad z \in \gamma \quad ,$$

So the range of $h(z)$ lies **inside** a circle center -1 and radius 1. This region does not include the origin $z = 0$, hence the **change of argument** is 0 (this is true for any countour whose inside does not including the origin. The "change of argument" is called the *winding number*, the number of times it winds around the **origin**. If the origin is outside the contour the winding number is of course 0.)

By the argument principle, the number of zeros of $h(z)$ (counting multiplicities) is the same as the number of poles (counting multiplicities). But the zeros of $h(z)$ are the zeros of its numerator, $g(z)$, while the poles of $h(z)$ are the zeros of its denominator, $f(z)$. **QED**.

Rouché's theorem is very good in proving that all the roots of a polynomial lie in a certain annulus.

Problem 20.2: Show that all the five zeros of

$$p(z) = z^5 + 2\,i\,z^2 + 2\,i\,z - 101 \quad ,$$

lie in the annulus $2 < |z| < 3$.

Solution of Problem 20.2: On the circle $|z| = 2$ (by the **triangle inequality**)

$$|p(z) + 101| = |z^5 + 2\,i\,z^2 + 2\,i\,z| \leq |z|^5 + 2|z|^2 + 2|z| = 2^5 + 2 \cdot 2^2 + 2 \cdot 2 = 32 + 8 + 4 = 44 < 101 \quad ,$$

So, by Rouché, $p(z)$ and $f(z) = 101$ have the same number of zeros inside the circle $|z| = 2$. But 101, being a constant, has no zeros, so neither does $p(z)$.

On the circle $|z| = 3$ we have, again by the triangle inequality

$$|p(z) - z^5| = |2\,i\,z^2 + 2\,i\,z - 101| \leq 2 \cdot |z|^2 + 2 \cdot |z| + 101 = 2 \cdot 3^2 + 2 \cdot 3 + 101 = 18 + 6 + 101 = 125 < 243 = |-z^5|$$

(since $3^5 = 243$).

So, by Rouché, $p(z)$ and $f(z) = z^5$ have the same number of zeros inside the circle $|z| = 3$. Of course $f(z) = -z^5$ has five zeros (counting multiplicity). Since any polynomial of degree n has altogether n roots (see the **Fundamental Theorem of Algebra**, coming up shortly) these are all of them.

So there are five roots in $|z| < 3$ and none of them in $|z| < 2$, so all five of them must be in the annulus $2 < |z| < 3$.

Rouché's theorem can also be used to prove one of the most important theorems in mathematics.

The Fundamental Theorem of Algebra: A polynomial of degree n has exactly n zeros, counting multiplicities.

Proof: WLOG the coefficient of z^n is 1 (such a polynomial is called a **monic** polynomial). So

$$p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 \quad ,$$

For **large** values of $|z| = R$,

$$\left| \frac{p(z) - z^n}{z^n} \right| = |a_{n-1}z^{-1} + \ldots + a_0 z^{-n}|$$

$$\leq \frac{|a_0| + \ldots + |a_{n-1}|}{R} \leq \frac{1}{2} \quad .$$

Hence $|p(z) - z^n| \leq \frac{1}{2}|z|^n < |-z^n|$ for $|z| = R$, so thanks to Monsieur le professeur Eugene Rouché $p(z)$ has the same number of zeros in $|z| \leq R$ as $f(z) = -z^n$, that is of course is n. QED.

Comment: The above is an example of a *formal proof*. An informal proof is to use the 'semi-formal' language of Lecture 19. When R id huge, the leading term, z^n dominates, so the value of our general polynomial is very close to $R^n e^{n\,i\,theta}$ so the argument is essentially $n\theta$. As we move, along the circle $|z| = R$, from a starting point, say $z = R$ back to it, moving along counter-clockwise, the argument increases by $n2\pi$. Dividing by 2π gives n.