Dr. Z.'s Intro to Complex Variable Lecture 19 Notes: The Zeros of an Analytic Function (Part I)

By Doron Zeilberger

Intuitively Obvious but Important Fact: If $f(z)$ is analytic in a domain D and there is an **infinite** sequence of numbers (alias points) $\{z_i\}_{i=1}^{\infty}$ inside D, such that $f(z_i) = 0$ for all of them **and** the sequence $\{z_i\}$ **converges** to a number z_0, then f is the **zero function**.

Explanation: Since there is a sequence of points z_i where f is zero, that gets closer-and-closer to z_0, it follows that all the derivatives of $f(z)$ at $z = z_0$ are zero, and hence its Taylor series is 0 and hence the function is identically zero in D.

Important Consequence: A **zero** of an analytic function is **isolated**. You can draw a (possibly tiny) circle around it where there are no other zeros.

By considering the reciprocal of $f(z)$, the same thing is true for **poles**.

If $z = z_0$ happens to be a zero of an analytic function $f(z)$ then of course, you can write it as $(z - z_0)g(z)$ where $g(z)$ is also analytic. In other words, you can divide $f(z)$ by $z - z_0$ and still get an analytic function. If $g(z_0) \neq 0$ then $z = z_0$ is called a **simple** zero, but if $g(z_0) = 0$ you can divide again by $z - z_0$. Sooner or later (after a finite number of trials) you will get something that is not 0 at z_0. The largest positive integer m such that you can write

$$f(z) = (z - z_0)^m g(z) \quad ,$$

for $g(z)$ analytic, (and of course then $g(z_0) \neq 0$), is called the **order** of the zero $z = z_0$, aka **multiplicity**.

Analogously, if $z = z_0$ is a pole of $f(z)$ then if $(z - z_0)f(z)$ is already analytic, then $z = z_0$ is called a **simple pole**. If it is not, then perhaps $(z - z_0)^2 f(z)$ is. The smallest positive integer m such that $(z - z_0)^m f(z)$ is analytic at $z = z_0$ is called the **order** of the pole.

Important Theorem (Theorem 1 on p. 173) Suppose that h is analytic in a domain D except at a finite number of poles. Let γ be a piecewise smooth positively oriented simple closed curve in D which does not pass through any pole or zero of h and whose inside lies in D. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}\, dz = (Number \quad of \quad Zeros \quad inside \quad \gamma) - (Number \quad of \quad Poles \quad inside \quad \gamma)$$

Each zero and pole is counted according to its **multiplicity** (aka order).

Explanation: This follows from the **Residue Theorem**, and the facts, proved at the beginning of Lecture 16, about the residues at zeros and poles of $h'(z)/h(z)$. If $z = z_0$ is a **zero** of order m, then the residue at z_0 is m, (a **positive** integer), and if $z = z_0$ is a **pole** of order n then the residue at z_0 is $-n$, (a **negative** integer).

Recall that every **non-zero** complex **number** z can be written in **polar representation**

$$z = re^{i\theta} \quad , \quad -\pi \leq \theta < \pi \quad ,$$

where $r = |z|$ is the **absolute value**, and θ is called the **Argument**. So we can write

$$z = |z|e^{i\,Arg(z)} \quad ,$$

However, since $e^{2i\pi} = 1$ it is also true that $z = |z|e^{i(Arg(z)+2\pi n)}$, for every integer n. Hence the **argument** in **lower case** has infinitely many choices.

(Note that the Argument of 0 is **undefined**).

Similarly, if $h(z)$ is any **function** then we can write (as long as $h(z) \neq 0$)

$$h(z) = |h(z)|e^{iarg(h(z))} \quad ,$$

Taking the log we have

$$\log\, h(z) = \log\, |h(z)| + iarg(h(z) \quad ,$$

But log as opposed to Log, is ambiguous (like arg as opposed to Arg), so it folllows, by a clever argument (see the book), that

$$\frac{1}{i}\int_{\gamma} \frac{h'(z)}{h(z)}\, dz$$

describes the **change of argument** of the function $h(z)$ as it travels, continuously along the contour γ. The formal proof (for the special case of a circle) is in pp. 174-175 of Fischer's book (but you are not expected to reproduce it).

This leads to the very **important**

Argument Principle (Theorem 2 on p. 176) Suppose that h is analytic in a domain D except at a finite number of poles. Let γ be a piecewise smooth positively oriented simple closed curve in D which does not pass through any pole or zero of h and whose inside lies in D. Then (**counting multiplicities**)

$$(Number \quad of \quad Zeros \quad inside \quad \gamma) - (Number \quad of \quad Poles \quad inside \quad \gamma)$$

equals the change of $arg\, h(z)$ as z travels along γ , divided by 2π.

In particular, if $h(z)$ is analytic (e.g. a polynomial), it has no poles, so to find the number of zeros (counting multiplicities) you just observe how the argument of $h(z)$ changes as z travels along the curve.

The argument principle is important both theoretically and practically. Theoretically, it implies, as we are going to see in the next lecture, the **Fundamental Theorem of Algebra**, that was open for many years, and that was first proved by Carl Friedrich Gauss. It is a theorem in **algebra**, but it uses complex analysis! Practically, it is a good way to locate regions where there are zeros, so that we have good starting points for Numerical methods like Newton-Raphson, and also to make sure that we got all the zeros in the region.

Let's illustrate the argument principle with a simple example, $h(z) = z^n$. The equation $h(z) = 0$ has only one root, namely $z = 0$, but it has **multiplicity** n, so 'the number of **roots** of the **equation** $z^n = 0$', (which is the same thing as the number of **zeros** of the **function** $h(z) = z^n$), **counting multiplicities** in any disc $|z| \le r$, is n.

Let's check the argument principle with this simple example. As we travel along the contour $|z| = r$, the parametric equation is $z = re^{i\theta}$, $-\pi \le \theta < \pi$, so the argument of $h(z) = r^n e^{ni\theta}$ is $n\theta$. At the starting point $z = r$ ($\theta = 0$), the argument is 0, since then $h(r) = r^n$ is real and positive so its Argument is 0. As it moves around the circle, at $\theta = \pi/n$ (i.e. at $z = re^{i\frac{\pi}{n}}$) the argument is π, alias $-\pi$, since $h(re^{i\frac{\pi}{n}}) = r^n re^{i\pi}$, so the argument at that point is π. At $z = r\, e^{2i\pi/n}$ $h(z) = r^n$ again, so the **Argument** is 0, since 2π is the same as 0 (but it is more convenient to use 'argument' and then it is 2π). So the range ("output") of the function $h(z) = z^n$ completed a **full circle** as the input z, moved along the segment of the circle $0 \le \theta \le \frac{2\pi}{n}$.

In the next segment, we have $\frac{2\pi}{n} \le \theta \le \frac{4\pi}{n}$, so it covers another full circle. This is repeated n times, so the total **change of argument**, until returning to the starting point, $z = r$, is n times 2π, i.e. $2\pi n$. By the argument principle, the number of zeros inside the contour is the change in argument divided by 2π, hence we get n.

IMPORTANT WARNING: Do not confuse **argument** and **Argument**. The Argument of a complex number, by **definition** is always in the half-open half-closed interval $[-\pi, \pi)$ i.e. $Arg(z)$ is the unique θ such that $z = re^{i\theta}$ and in addition $-\pi \le \theta < \pi$. But there are **infinitely many** θ such that $z = |z|\, e^{i\theta}$, namely $\theta = Arg(z) + 2\pi n$, for any integer n.

When we determine the **change of argument** of a function $h(z)$ as it transverses (in the positive direction) a contour γ, we use *argument* not *Argument*, since it often passes the "**International date line**" $Arg(z) = -\pi$ several times, and we do not "adjust the clock". Whenever we transverse a contour γ starting at a point and returning to the beginning, the "change in Argument" is always 0, so it is not very informative.

Let's illustrate the argument principle with another simple example.

Problem 19.1: Use the **argument principle** (no credit for other methods) to find the number of roots of the equation $z^2 - i = 0$ that lie in the first quadrant, $\{z : Re\, z > 0, Im\, z > 0\}$.

Sol. of 19.1: Of course for this very simple example, we can do it directly, since we can actually find all the roots of the equations $z^2 = i$. Let's do it just for fun (and for checking).

Writing (as we learned, way back in the good old, *non-remote* classroom days) $z = re^{i\theta}$, and

$i = e^{i\frac{\pi}{2}}$, we have

$$r^2 e^{2i\theta} \ = \ e^{i\frac{\pi}{2}}$$

Hence

$$r = 1 \quad and \quad 2\theta = \frac{\pi}{2} + 2n\pi \quad , (n = 0, 1) \quad so$$

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4} \quad ,$$

and the two roots are

$$e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \quad , \quad e^{i\frac{5\pi}{4}} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \quad .$$

So we know that there is exactly **one** root of the equation $z^2 - i = 0$ in the *first* quadrant, exactly **one** root in the *third* quadrant, and **none** in the *second* and *fourth* quadrants.

Let's establish it using the argument principle. The same method can be used to tackle much more complicated polynomials (and functions) for which it is not so easy to find the zeros.

Consider the quarter-circle, center the origin, of radius R, that lies in the first quadrant, where R is VERY BIG. Starting at the origin $z = 0$ let's transverse it in the **positive** (counter-clockwise) direction. It has three segments:

• **First segment**: Moving along the real (alias x) positive axis from $z = 0$ to $z = R$ (alias from $(0, 0)$ to $(R, 0)$)

• **Second segment**: Moving along the arc $\{z = Re^{i\theta} \ : \ 0 \leq \theta \leq \frac{\pi}{2}\}$ from $z = R$ (alias $(R, 0)$) to $z = iR$ (alias $(0, R)$)

• **Third segment**: Moving down the imaginary (alias y) positive axis from $z = iR$ back to $z = 0$ (alias from $(0, R)$ to $(0, 0)$).

Let's examine the **change in argument** in each segment separately, and at the end, we will add them up.

In the **first segment**, it starts out being $f(0) = 0^2 - i = -i = e^{-i\frac{\pi}{2}}$, and its **Argument** is $-\frac{\pi}{2}$. (At first we take the principal argument, i.e. the Argument, that, by convention lies in $[-\pi, \pi)$, but later on we would have to take other choices, since the argument of a continuous function is continuous). At the end of that segment $f(R) = R^2 - i$ i.e. the point $(R^2, -1)$. Since R is HUGE (you can thing of it as practically infinite), this lies very close to the positive real axis, and hence the argument at the end of the first segment is 0.

Hence: Net Gain of argument due to going along the **first segment** is $0 - (-\frac{\pi}{2}) = \frac{\pi}{2}$.

In the **second segment**, it starts out with argument 0 (as we saw above). Now $f(Re^{i\theta}) = R^2 e^{i(2\theta)} - i$ is practically the same as $R^2 e^{i(2\theta)}$. The argument is 2θ, so as z moves along the circular arc $0 \leq \theta \leq \frac{\pi}{2}$, the net gain is $2\frac{\pi}{2} - 2 \cdot 0 = \pi$.

Hence: Net Gain of argument due to going along the **second segment** is π.

In the **third segment**, it starts out with argument π (as we saw above), **alias** $-\pi$ (we crossed the "international date line"), and when we go down the y axis $f(iy) = -y^2 - i$ travels along the third quadrant until it reaches the origin and then the argument, as we sat at the very beginning of the journey is $-\frac{\pi}{2}$.

Hence: Net change of argument due to going along the **third segment** is $-\frac{\pi}{2} - (-\pi) = \frac{\pi}{2}$.

Combining the three changes we have

Total change in argument traveling along our contour $= \frac{\pi}{2} + \pi + \frac{\pi}{2} = 2\pi$.

Dividing by 2π, we get (not surprisingly) that

Ans. to 19.1: the number of zeros of $f(z) = z^2 - i$ in the first quadrant is 1.

Note that in the above example, all the changes were positive. Sometimes, some of the segments give you negative contributions. So let's do the following problem.

Problem 19.2: Use the **argument principle** (no credit for other methods) to find the number of roots of the equation $z^2 - i = 0$ that lie in the second quadrant, $\{z : Re\, z < 0, Im\, z > 0\}$.

Sol. of 19.2: We know (by doing it directly) that the answer should be zero. Let's check it, and see whether the argument principle agrees.

Consider the quarter-circle, center the origin, of radius R, that lies in the second quadrant, where R is VERY BIG. Starting at the origin $z = 0$ let's transverse it in the **positive** (counter-clockwise) direction. It has three segments:

• **First segment**: Moving up along the imaginary (alias y) positive axis from $z = 0$ to $z = Ri$ (alias from $(0, 0)$ to $(0, R)$)

• **Second segment**: Moving along the arc $\{z = Re^{i\theta} : \frac{\pi}{2} \leq \theta \leq \pi\}$ from $z = Ri$ (alias $(0, R)$) to $z = -R$ (alias $(-R, 0)$)

• **Third segment**: Moving left on the real (alias x) negative axis from $z = -R$ back to $z = 0$ (alias from $(-R, 0)$ to $(0, 0)$).

Let's examine the **change in argument** in each segment separately, and at the end, we will add them up.

In the **first segment**, it starts out being $f(0) = 0^2 - i = -i = e^{-i\frac{\pi}{2}}$, and its **Argument** is $-\frac{\pi}{2}$. At the end of the segment $f(iR) = -R^2 - i$ i.e. the point $(-R^2, -1)$. Since R is HUGE (you can thing of it as practically infinite), this lies very close to the negative real axis, and hence the argument at the end of the first segment is $-\pi$.

Hence: Net Gain of argument due to going along the **first segment** is $-\pi - (-\frac{\pi}{2}) = -\frac{\pi}{2}$.

Comment: Note that in this segment the change in argument is **negative**. It is like **retrograde movements of planets** in the night sky.

In the **second segment**, it starts out with argument $-\pi$ (as we saw above). As before, $f(Re^{i\theta}) = R^2e^{i(2\theta)} - i$ is practically the same as $R^2e^{i(2\theta)}$. The argument is 2θ, increasing from $\theta = \frac{\pi}{2}$ to $\theta = \pi$, so the net gain is π.

Hence: Net Gain of argument due to going along the **second segment** is π.

In the **third segment**, from $z = -R$ to $z = 0$ it starts out at $f(-R) = R^2 - i$ that is practically pointing right, so the argument is 0 and then since $f(-x) = x^2 - i$, stays in the fourth quadrant, the argument goes down from being 0 to being $-\frac{\pi}{2}$.

Hence: Net change of argument due to going along the **third segment** is $(-\frac{\pi}{2}) - 0 = -\frac{\pi}{2}$.

Combining the three changes we have:

Total change in argument traveling along our contour equals $-\frac{\pi}{2} + \pi - \frac{\pi}{2} = 0$.

Dividing by 2π, we get that the number of zeros of $f(z) = z^2 - i$ in the second quadrant is 0.

Now let's tackle a much more complicated problem.

Problem 19.3: Use the **argument principle** (no credit for other methods) to find the number of zeros of the equation $f(z) = z^9 - iz^5 + 100i$ that lie in the first quadrant, $\{z : Re\, z > 0, Im\, z > 0\}$.

Sol. of 19.3: Consider the quarter-circle, center the origin, of radius R, that lies in the first quadrant, where R is VERY BIG. Starting at the origin, $z = 0$, let's transverse it in the **positive** (counter-clockwise) direction. It has three segments:

• **First segment**: Moving along the real (alias x) positive axis from $z = 0$ to $z = R$ (alias from $(0,0)$ to $(R,0)$)

• **Second segment**: Moving along the arc $\{z = Re^{i\theta} : 0 \leq \theta \leq \frac{\pi}{2}\}$ from $z = R$ (alias $(R,0)$) to $z = iR$ (alias $(0,R)$)

• **Third segment**: Moving down the imaginary (alias y) positive axis from $z = iR$ back to $z = 0$ (alias from $(0,R)$ to $(0,0)$).

Let's examine the **change in argument** in each segment separately, and at the end, we will add them up.

In the **first segment**, it starts out being $f(0) = 100i$, and its **Argument** is $\frac{\pi}{2}$. A typical value of $f(z)$ along this segment is $x^9 - ix^5 + 100i$ that is the point $(x^9, 100 - x^5)$ that for large R is

$(R^9, 100 - R^5)$, and its argument arctan $\frac{100-R^5}{R^9}$ tends to $\arctan(0) = 0$. so the argument goes down to 0, and at 'infinity' equals 0.

Hence: Net Gain of argument due to going along the **first segment** is $0 - \frac{\pi}{2} = -\frac{\pi}{2}$.

In the **second segment**, it starts out with argument 0 (as we saw above). Now $f(Re^{i\theta})$ is approximately $R^9 e^{i(9\theta)}$ (only the leading term matters). The argument is 9θ, so as z moves along the circular arc $0 \leq \theta \leq \frac{\pi}{2}$, the net gain is $9\frac{\pi}{2}$.

Hence: Net Gain of argument due to going along the **second segment** is $\frac{9\pi}{2}$.

In the **third segment**, it starts out with argument $\frac{9\pi}{2}$ **alias** $\frac{\pi}{2}$ and since $f(0) = 100i$ has argument $\frac{\pi}{2}$, the argument does not change along this segment.

Hence: Net change of the argument due to going along the **third segment** is 0.

Combining the three changes we have

Total change in argument traveling along our contour $= -\frac{\pi}{2} + \frac{9\pi}{2} + 0 = 4\pi$.

Dividing by 2π, we get

Ans. to 19.3: The number of zeros of $f(z) = z^9 - iz^5 + 100i$ in the first quardrant is 2.

The next two examples concern the number of zeros in **half-planes**.

Problem 19.4: Use the **argument principle** (no credit for other methods) to find the number of zeros of the equation $f(z) = z^7 - i$ that lie in the upper half plane $\{z : Im\, z > 0\}$.

Solution to 19.4: We now consider the half-circle

$$\{z = Re^{i\theta} \,:\, 0 \leq \theta \leq \pi\} \cup \{z = x + i0 \,:\, -R < x < R\} \quad .$$

Now there are only two pieces. First let' go (in the **rightbound direction**) from $z = -R$ to $z = R$ along the real axis. Since $f(-R)$ is essentially $-R^7$ so its argument is the same as -1, i.e. $-\pi$. On the other hand $f(R)$ is essentially R^7 so its argument is the same as 1, in other words 0. The net gain to the argument of $f(z)$ along that segment (i.e. along the x axis from $(-R, 0)$ to $(R, 0)$ is $0 - (-\pi) = \pi$.

Going along the circular arc, $f(Re^{i\theta})$ is essentially $R^7 e^{7i\theta}$ so the argument goes up by 7π. Together the gain is $\pi + 7\pi = 8\pi$ and dividing by 2π gives 4.

Ans. to 19.4: The function $f(z) = z^7 - i$ has 4 roots in the upper half-plane $\{z : Im\, z > 0\}$.

Problem 19.5: Use the **argument principle** (no credit for other methods) to find the number of zeros of the equation $f(z) = z^7 - i$ that lie in the lower half plane $\{z : Im\, z < 0\}$.

Solution to 19.5: We now consider the lower half-circle

$$\{z = Re^{i\theta} \ : \ -\pi \leq \theta \leq 0\} \cup \{z = x + i0 \ : \ -R < x < R\} \quad .$$

Again there are only two pieces. Now we go from $z = R$ to $z = -R$ along the real axis (we have to go from right to left, rather than from left to right as in the previous problem, since the movement is always counter-clockwise).

$f(R)$ is essentially R^7 so its argument is the same as 1, i.e. 0. On the other hand $f(-R)$ is essentially $-R^7$ so its argument is the same as -1, in other words $-\pi$. The net gain to the argument of $f(z)$ along that segment (i.e. along the x axis from $(R, 0)$ to $(-R, 0)$ is $-\pi - 0 = -\pi$.

Going along the circular arc, $f(Re^{i\theta})$ is essentially $R^7 e^{7i\theta}$ going from $\theta = -\pi$ to $\theta = 0$. So the argument goes up by 7π. Combining the contributions from the two pieces, the total gain is $-\pi + 7\pi = 6\pi$ and dividing by 2π gives 3.

Ans. to 19.5: The function $f(z) = z^7 - i$ has 3 roots in the lower half-plane $\{z : Im\, z < 0\}$.