Dr. Z.'s Intro to Complex Variable Lecture 18 Notes

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Evaluating Complicated Trig Integrals

In Lecture 12 (section 2.3) we learned how to to use **complex** variable methods to compute **real** trig integrals like the next problem.

Problem 18.1:

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} \quad .$$

Let's review how we did it.

Solution to Problem 18.1 (old way)

First we used the famous formula $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, then we wrote $z = e^{i\theta}$, so $\cos\theta = \frac{z+z^{-1}}{2}$. Since $dz = ie^{i\theta}d\theta$ we have $d\theta = \frac{dz}{iz}$ and the real trig integral became the complex **contour integral**

$$\int_{|z|=1} \frac{\frac{dz}{iz}}{2+\frac{z+z^{-1}}{2}} \; = \; -i \int_{|z|=1} \frac{2}{z^2+4z+1}\, dz \quad .$$

This is a **rational function** where the bottom is quadratic. Writing

$$z^2+4z+1 = (z-p)(z-q) \quad ,$$

where

$$p,q \; = \; \frac{-4 \pm \sqrt{4^2-4\cdot 1\cdot 1}}{2} = \frac{-4\pm\sqrt{12}}{2} = \frac{-4\pm 2\sqrt{3}}{2} = -2\pm\sqrt{3} \quad ,$$

we get $p = -2+\sqrt{3}$ and $q = -2-\sqrt{3}$. Note that $|p| < 1$ and $|q| > 1$, so p is **inside** the unit circle $|z| = 1$ and q is **outside**.

Now we write the contour integral as

$$-i \int_{|z|=1} \frac{2}{z^2+4z+1}\, dz = -i \int_{|z|=1} \frac{2}{(z-p)(z-q)}\, dz \; = \; -i \int_{|z|=1} \frac{\frac{2}{(z-q)}}{z-p}\, dz \quad .$$

Recall that the famous **Cauchy Formula** that says that if $F(z)$ is analytic in a domain D and γ is a closed curve (contour) and $z = p$ is **inside** the curve then

$$\int_\gamma \frac{F(z)}{z-p} \; = \; 2\pi i F(p) \quad .$$

In this case $F(z) = \frac{2}{z-q}$ is analytic, since its only "trouble-maker" is outside the curve, so the answer is

$$(-i)(2\pi i)\frac{2}{p-q} \quad .$$

Since $p - q = (-2+\sqrt{3}) - (-2-\sqrt{3}) = 2\sqrt{3}$, we get that our trig integral is.

$$(-i)(2\pi i)\frac{2}{2\sqrt{3}} \; = \; -2i^2\pi\frac{1}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}} = \frac{2\pi\sqrt{3}}{3} \quad .$$

A quicker way

Before going on let's state

VERY IMPORTANT SHORTCUT FOR COMPUTING THE RESIDUE AT A SIMPLE POLE

If $f(z) = \frac{P(z)}{Q(z)}$ is a **rational function** and z_0 is a **simple pole**, then

$$Res(f(z); z_0) = \frac{P(z_0)}{Q'(z_0)} \quad .$$

Warning: Only applicable for **simple** poles.

The proof is easy. The residue is $\lim_{z\to z_0} \frac{(z-z_0)P(z)}{Q(z)}$, where both top and bottom are 0 at $z = z_0$. Using L'Hôpital, this equals $\lim_{z\to z_0} \frac{((z-z_0)P(z))'}{Q'(z)}$ that equals $\lim_{z\to z_0} \frac{(z-z_0)P'(z)+P(z)}{Q'(z)}$. Now plug-in $z = z_0$.

Solution to Problem 18.1 (new way)

We can do the above integral faster, using the powerful **Residue Theorem**. Recall that it says that if the only singularties of $F(z)$ **inside** the contour γ are **poles** , let's call them $z_1, \ldots, z_k$, then

$$\int_\gamma f(z)\, dz \; = \; 2\pi i \left(\sum_{i=1}^{k} Res(f(z); z_i) \right) \quad .$$

The function $\frac{2}{z^2+4z+1}$ only has one pole, $z = p$, so

$$Res(\frac{2}{z^2+4z+1}\, ; p) = \frac{2}{2p+4} = \frac{2}{2(-2+\sqrt{3})+4} = \frac{2}{-4+2\sqrt{3}+4} = \frac{1}{\sqrt{3}} \quad ,$$

so the integral that we need is $(-i)(2\pi i)\frac{1}{\sqrt{3}} = \frac{2\pi\sqrt{3}}{3}$, as before.

Comment: In some sense, Cauchy's formula is a special case of the Residue theorem where there is only one pole inside the contour, and that pole happens to be simple.

Equipped with the Residue theorem, we can compute much more complicated integrals.

Problem 18.2: Compute the following integral

$$\int_0^{2\pi} \frac{d\theta}{3+\cos^2\theta}$$

Sol. to 18.2: We use the fact that $\cos\theta = (z+z^{-1})/2$ where $z = e^{i\theta}$, so $dz = ie^{i\theta}d\theta = izd\theta$, so $d\theta = \frac{dz}{iz}$.

The integral becomes the **contour integral** over the unit circle $|z| = 1$.

$$\frac{1}{i}\int_{|z|=1} \frac{dz}{(3+(z+1/z)^2/4)z} = \frac{1}{i}\int_{|z|=1} \frac{4\,dz}{(12+(z+1/z)^2)z} = \frac{1}{i}\int_{|z|=1} \frac{4\,dz}{(12+z^2+2+z^{-2})z} =$$

$$\frac{1}{i}\int_{|z|=1} \frac{4\,dz}{(z^2+14+z^{-2})z} = \frac{1}{i}\int_{|z|=1} \frac{4z\,dz}{(z^4+14z^2+1)} \quad .$$

$\frac{4z\,dz}{(z^4+14z^2+1)}$ is a **rational function**. Let's find its poles. Putting $z^2 = X$ we get the quadratic equation

$$X^2+14X+1=0 \quad ,$$

whose roots are

$$\frac{-14 \pm \sqrt{14^2-4\cdot 1\cdot 1}}{2\cdot 1} = \frac{-14\pm\sqrt{192}}{2} = -7\pm\sqrt{48} \quad .$$

So there are two roots $-7-\sqrt{48}$ whose absolute value is more than 1 and $-(7-\sqrt{48})$ whose absolute value is less than 1.

Going back to the roots of $z^4 + 14z^2 + 1 = 0$, the two roots that are inside the unit circle , and hence the poles of the integrand that are in $|z| < 1$ are

$$z_1 = \sqrt{7-\sqrt{48}}\; i \quad , \quad z_2 = -\sqrt{7-\sqrt{48}}\; i \quad ,$$

(Note that $z_1^2 = z_2^2 = -(7-\sqrt{48})$).

We need the residues of $\frac{4z}{z^4+14z^2+1}$ at $z = z_1$ and $z = z_2$.

Recall that the residue of a rational function $P(z)/Q(z)$ at a simple pole $z = z_1$, is $P(z_1)/Q'(z_1)$. Here $P(z) = 4z$ and $Q(z) = z^4 + 14z^2 + 1$ so $Q'(z) = 4z^3 + 28z$ and it follows that the residue is $\frac{4z_1}{4z_1^3+28z_1} = \frac{1}{z_1^2+7}$

But $z_1^2 = \sqrt{48} - 7$ so $z_1^2 + 7 = \sqrt{48}$ and the residue is $\frac{1}{\sqrt{48}}$. Similarly, the residue at $z = z_2$ is also $\frac{1}{\sqrt{48}}$.

It follows that our integral equals

$$(2\pi i)\frac{1}{i}\left(\frac{1}{\sqrt{48}}+\frac{1}{\sqrt{48}}\right) = \frac{4\pi}{\sqrt{48}} = \frac{\pi}{\sqrt{3}} = \frac{\pi\sqrt{3}}{3} \quad .$$

Ans. to 18.2: $\int_0^{2\pi} \frac{d\theta}{3+\cos^2\theta} = \frac{\pi\sqrt{3}}{3}$.

Here is another integral of this kind.

Problem 18.3 Compute the following integral

$$\int_0^{2\pi} \frac{d\theta}{(3+\cos\theta)(2+\cos\theta)} \quad .$$

Solution to 18.3: We can use the above method directly, but it would be rather complicated. It is more efficient to do some **pre-processing**, pretend that $\cos\theta$ is X and use partial fractions

$$\frac{1}{(3+X)(2+X)} = \frac{A}{3+X} + \frac{B}{2+X} \quad ,$$

getting

$$A(2+X)+B(3+X)=1$$

plugging-in $X=-2$ gives $B=1$ and X=-3 gives $A=-1$ so the integrand is

$$\frac{1}{(3+\cos\theta)(2+\cos\theta)} = \frac{1}{2+\cos\theta} - \frac{1}{3+\cos\theta}.$$

Hence

$$\int_0^{2\pi} \frac{d\theta}{(3+\cos\theta)(2+\cos\theta)} = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} - \int_0^{2\pi} \frac{d\theta}{3+\cos\theta} \quad . \quad .$$

The first piece is **exactly** Problem 18.1,

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi\sqrt{3}}{3} \quad .$$

and the second piece is done the same way. I will let you do it yourself, after the lecture. You should get:

$$\int_0^{2\pi} \frac{d\theta}{3+\cos\theta} = \frac{\pi\sqrt{2}}{2} \quad .$$

Combining, we get

Ans. to 18.3:

$$\int_0^{2\pi} \frac{d\theta}{(3+\cos\theta)(2+\cos\theta)} = \pi\left(\frac{2\sqrt{3}}{3} - \frac{\sqrt{2}}{2}\right) \quad .$$

Here is an even more complicated integral.

Problem 18.4: Compute the following integral

$$\int_0^{2\pi} \frac{d\theta}{(3+\cos^2\theta)(2+\cos^2\theta)} \quad .$$

Solution to 18.4: Once again, we can do it directly, but it would be rather complicated. It is more efficient to do some **pre-processing**, pretend that $\cos^2\theta$ is X and use partial fractions

$$\frac{1}{(3+X)(2+X)} = \frac{A}{3+X} + \frac{B}{2+X} \quad ,$$

getting

$$A(2+X) + B(3+X) = 1$$

plugging-in $X = -2$ gives $B = 1$ and X=-3 gives $A = -1$, so the integrand is

$$\frac{1}{(3+\cos^2\theta)(2+\cos^2\theta)} = \frac{1}{2+\cos^2\theta} - \frac{1}{3+\cos^2\theta} .$$

Hence

$$\int_0^{2\pi} \frac{d\theta}{(3+\cos^2\theta)(2+\cos^2\theta)} = \int_0^{2\pi} \frac{d\theta}{2+\cos^2\theta} - \int_0^{2\pi} \frac{d\theta}{3+\cos^2\theta} \quad . \quad .$$

The second piece is **exactly** Problem 18.2

$$\int_0^{2\pi} \frac{d\theta}{3+\cos^2\theta} = \frac{\pi\sqrt{3}}{3} \quad .$$

and the first piece is done the same way. I will let you do it yourself, after the lecture. You should get:

$$\int_0^{2\pi} \frac{d\theta}{2+\cos^2\theta} = \frac{\pi\sqrt{6}}{3} \quad .$$

Combining, we get

Ans. to 18.4:

$$\int_0^{2\pi} \frac{d\theta}{(3+\cos^2\theta)(2+\cos^2\theta)} = \pi\left(\frac{\sqrt{6}-\sqrt{3}}{3}\right) \quad .$$