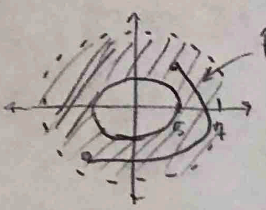
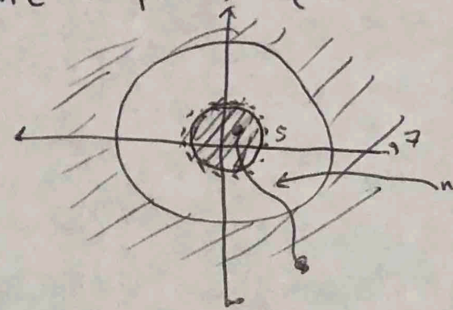
Jason Luo → Final Exam Work

1. $R = \{z : 5 \le |z| < 7\}$ → $\mathbb{C} \setminus R$ (complement) $\{z : |z| < 5 \text{ or } |z| \ge 7\}$





2. a) using definition of analytic function, for some $z \in \mathbb{C}$

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{(\bar{z} + \bar{h})^2 - (\bar{z})^2}{h}$$

→ Approaching along the imaginary axis, $h = ik$ $(k \to 0)$:

$$\lim_{k \to 0} \frac{(\bar{z} - ik)^2 - (\bar{z})^2}{ik} = \frac{\cancel{\bar{z}^2} - 2\bar{z}ik + (ik)^2 - \cancel{(\bar{z})^2}}{ik} = -2\bar{z} + ik \to -2\bar{z}$$

— Approaching along the real axis, $h = k$ $(k \to 0)$:

$$\lim_{k \to 0} \frac{(\bar{z} + k)^2 - (\bar{z})^2}{k} = \frac{\cancel{\bar{z}^2} + 2\bar{z}k + k^2 - \cancel{\bar{z}^2}}{k} = 2\bar{z} + k \longrightarrow 2\bar{z}$$

In general $2\bar{z} \ne -2\bar{z}$, so the answer is different depending on the direction of approach, and f is not analytic.

b) using the Cauchy-Riemann equations

$f(x+iy) = (x - iy)^2 = x^2 - 2ixy + (iy)^2 = x^2 - y^2 - 2xy\,i$

$u = x^2 - y^2 \qquad v = -2xy$

Observe that $\frac{\partial u}{\partial y} = -2y$, and $\frac{-\partial v}{\partial x} = -(-2y) = 2y$

In general $-2y \ne 2y$, so $\frac{\partial u}{\partial y} \ne \frac{-\partial v}{\partial x}$ and the Cauchy-Riemann equations fail to hold. Thus f is not analytic.

3. Find the roots of the quadratic equation $z^2-(2+2i)z+2i=\emptyset$

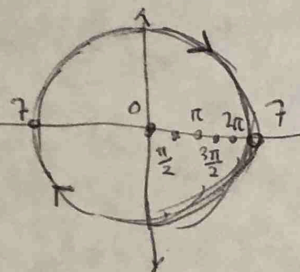
$z^2 - 2z - 2iz + 2i =$ Use the quadratic equation

$$z = \frac{(2+2i) \pm \sqrt{(2+2i)^2 - 4(2i)}}{2} = 1+i \pm \sqrt{\cancel{4}+\cancel{8i}+\cancel{4i^2}-\cancel{8i}} = 1+i$$

only one root, $z=1+i$ → Checking, $(z-1-i)^2 = z^2-2(1+i)z+(1+i)^2 =$
$z^2-(2+2i)z+\cancel{1}+2i-\cancel{1}$, which recovers the original equation ✓.

4. $\cot z = \frac{\cos z}{\sin z}$ → Zeros = zeros of $\cos z$, poles = zeros of $\sin z$

$(n+\frac{1}{2})\pi$ $\qquad$ $n\pi$



→ the largest n such that $\frac{n\pi}{2} < 7$ is $n=4$. Thus, the roots of $\cos z$ inside $|z|<7$ are:

$z = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}$ (all order 1)

roots of $\sin z$ are:

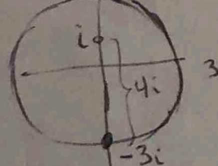
$z = \emptyset, \pi, -\pi, 2\pi, -2\pi$ (all order 1)

Thus, change in argument is: $2\pi(\#\text{zeros} - \#\text{poles}) = 2\pi(4-5) = -2\pi$

BUT, we are travelling clockwise, so we have to flip the sign, yielding $\boxed{2\pi}$

5. a) max $|z-i|^3$ on $\{z:|z|\le 3\}$ and its locations.

Since $(z-i)^3$ is analytic (entire, even) & nonconstant, its max. mod. must lie on the boundary (by maximum modulus principle.) $|z|=3$. Then $|z-i|^3$ is maximized when $|z-i|$ is maximized (since x^3 is increasing everywhere), or when z is furthest from i. On the boundary $|z|=3$, this occurs at $\boxed{z=-3i}$ → then $|z-i|^3 = |-4i|^3 = 4^3 = \boxed{64}$



b) min $|z-i|^3$ on $\{z:|z|\le 3\}$. The minimum value possible for $|X|$ in general is $\emptyset$ (by positive-definiteness of the modulus). This is attained at $z=i$ (which lies in $|z|\le 3$), since $|i-i|^3 = |0|^3 = 0$. Thus this must be the minimum value possible. The minimum can only occur at $z=i$ since $|z-i|^3=0 \Rightarrow |z-i|=0$

$\Rightarrow z-i=0 \Rightarrow z=i$

6. a) $\int_{0\to 1} z\,dz \qquad z(t) = t \rightarrow \quad dz = dt \quad (0 \leq t \leq 1)$

$$\rightarrow \int_0^1 t\,dt = \frac{t^2}{2}\Big|_0^1 = \boxed{\frac{1}{2}}$$

b) $\int_{1\to i} z\,dz \qquad z(t) = 1(1-t)+it = 1+(i-1)t \qquad (0 \leq t \leq 1)$

$$\hookrightarrow dz = (i-1)\,dt$$

$$\rightarrow \int_0^1 (1+(i-1)t)(i-1)\,dt = \int_0^1 i-1+(i-1)^2 t = (i-1)t + \frac{(i-1)^2 t^2}{2}\Big|_0^1$$

$$= i-1+\frac{\cancel{1}-2i+\cancel{-1}}{2} = i-1-i = \boxed{-1}$$

c) $\int_{i\to 0} z\,dz \qquad z(t) = i - it \quad (0 \leq t \leq 1)$

$$\hookrightarrow dz = -i\,dt$$

$$\rightarrow \int_0^1 (i-it)(-i)\,dt = \int_0^1 (1-t)\,dt = t - \frac{t^2}{2}\Big|_0^1 = 1-\frac{1}{2} = \boxed{\frac{1}{2}}$$

d) $\boxed{\frac{1}{2} - 1 + \frac{1}{2} = 0}$ ✓

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7. Compute $\displaystyle\int_{|z|=5} \frac{(z-1)(z-2)}{(z-1-i)(z-4i)}\,dz$ poles: $\underbrace{z = 1+i,\ 4i}_{\text{both lie in the contour } |z|<5}$ inside of the

Residue @ $z=1+i \rightarrow$

$$\frac{(\cancel{1}+i-\cancel{1})(1+i-2)}{(1+i-4i)} = \frac{i(-1+i)}{(1-3i)} = \frac{-1-i}{(1-3i)}\frac{(1+3i)}{(1+3i)} = \frac{-1-3i-i+3}{10}$$

$$= \frac{2-4i}{10}$$

Residue @ $z=4i \rightarrow$

$$\frac{(4i-1)(4i-2)}{(4i-1-i)} = \frac{-16-8i-4i+2}{(-1+3i)} = \frac{(+14+12i)}{-1+3i}\frac{(+1+3i)}{(-1-3i)} = \frac{14+42i+12i-36}{10}$$

$$= \frac{-22+54i}{10}$$

By the Residue Thm, the integral evaluates to $2\pi i\left(\frac{2-4i-22+54i}{10}\right) =$

$$2\pi i\left(\frac{-20+50i}{10}\right) = 2\pi i(-2+5i) = -4\pi i - 10\pi$$

$$\boxed{-10\pi - 4\pi i}$$

8) a) $\int_0^{2\pi} \cos\left(\frac{\pi}{3} + 3e^{it}\right) dt$ ↗ $\cos z$ is entire, so MVT applies, & this equals

$2\pi \cos\left(\frac{\pi}{3}\right) = 2\pi\left(\frac{1}{2}\right) = \boxed{\pi}$

b) $\int_0^{2\pi} \tan\left(\frac{\pi}{4} + 3e^{it}\right) dt$ $\tan z$ has a poles when $\cos z = 0$, ex. @ $z = \frac{\pi}{2}$

but $\left|\frac{\pi}{4} - \frac{\pi}{2}\right| = \left|\frac{\pi}{4}\right| < 3$, so $\tan z$ is not analytic

in the domain $\left|z - \frac{\pi}{4}\right| \leq 3$. Thus MVT is not applicable

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9. a) Show $\sum_{n=0}^{\infty} \frac{(i\pi/2)^n}{n!}$ converges. Use the ratio test:

$$\lim_{n\to\infty} \left| \frac{\left(\frac{i\pi}{2}\right)^{n+1}}{(n+1)!} \cdot \frac{n!}{\left(\frac{i\pi}{2}\right)^n} \right| = \frac{\left|\frac{i\pi}{2}\right|}{|n+1|} = \frac{\pi}{2(n+1)} \to 0 < 1.$$ $\boxed{\text{Thus the series converges}}$

b) determine its value. $\sum_{n=0}^{\infty} \frac{(i\pi/2)^n}{n!}$ has the form of the power series for e^z with $z = i\pi/2$,

so the value is

$$e^{\left(\frac{i\pi}{2}\right)} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = \boxed{i}$$

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10. $f(z) = z^2$ on $\{z : |z| < 1\}$

a) Writing in polar form, let $z = re^{i\theta}$. $|z| = |re^{i\theta}| = |r| < 1$.

Then $f(z) = r^2 e^{i2\theta}$ by De Moivre's Thm, and $|f(z)| = |r^2 e^{i2\theta}| = r^2$.

But $r^2 < 1$ if and only if $r < 1$ (assuming $r \geq 0$). Thus

$|z| < 1$ iff $|f(z)| < 1$, and D is mapped precisely onto itself $\boxed{\{z : |z| < 1\}}$

b) f is not one-to-one. Consider $z_1 = \frac{1}{2}$ & $z_2 = -\frac{1}{2}$. Clearly $z_1 \neq z_2$ &

$|z_1|, |z_2| < 1$ (so both are in D), but

$f(z_1) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ & $f(z_2) = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$, so $f(z_1) = f(z_2)$

11. Use Taylor poly of degree 2 at $z_0 = 2i$ of z^3 to approximate $(2i + \frac{1}{10})^3$

↳ since z^3 is a polynomial itself, we don't have to worry about convergence.

$f(2i) = (2i)^3 = -8i$

$f'(2i) \rightarrow f'(z) = 3z^2 \rightarrow 3(2i)^2 = -12$

$f''(2i) \rightarrow f''(z) = 6z \rightarrow 6(2i) = 12i$

Thus the Taylor polynomial of degree 2 is

$f \approx -8i - 12(z-2i) + 6i(z-2i)^2$

Thus, the polynomial evaluates to

At $z = 2i + \frac{1}{10}$, $(z-2i) = \frac{1}{10}$.

$(2i + \frac{1}{10})^3 \approx -8i - 12(\frac{1}{10}) + 6i(\frac{1}{10})^2 = -\frac{6}{5} + i(-8 + \frac{6}{100}) = \frac{-6}{5} - i\frac{397}{50}$

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12. Find the zeroes of $f(z) = z^3 - 1 - i$ lying ~~the~~ in the first quadrant $\{z = x+iy : x > 0, y > 0\}$

$z^3 - 1 - i = 0 \rightarrow z^3 = 1 + i$, we want the cube roots of $1+i$. Writing in polar form,

$1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2} \text{ cis } \frac{\pi}{4}$. If we let $z = re^{i\theta}$

we obtain $z^3 = r^3 e^{i3(\theta)}$, so $r^3 = \sqrt{2} \rightarrow r = 2^{1/6}$

Arguments (3 solution)

$3\theta = \frac{\pi}{4} \rightarrow \theta = \frac{\pi}{12} \rightarrow$ inside $0 < \theta < \frac{6\pi}{12}$

$3(\theta) = \pi/4 + 2\pi \rightarrow \theta = \frac{\pi}{12} + \frac{2\pi}{3} = \frac{9\pi}{12}$

$3(\theta) = \pi/4 + 4\pi \quad \theta = \frac{\pi}{12} + \frac{4\pi}{3} = \frac{17\pi}{12}$

outside $0 < \theta < \frac{6\pi}{12}$

To be in the 1st quadrant, the argument must be between $0 < \theta < \frac{\pi}{2} \Rightarrow 0 < \theta < \frac{6\pi}{12}$

Thus, only one root, $2^{1/6} e^{i(\frac{\pi}{12})}$ lies in the 1st quadrant

$$\boxed{z = 2^{1/6}\cos\left(\frac{\pi}{12}\right) + i2^{1/6}\sin\left(\frac{\pi}{12}\right)}$$

13) a) $\int_{\gamma_R} \frac{\cos z}{100+z^2}\,dz$ does not help because $|\cos z|$ is ~~the~~ unbounded* on the upper half-plane, so the "semi-circle arc $|z|=R$ above the x-axis"'s corresponding contour integral does not vanish as $R \to \infty$.

* Observe that $\cos(x+iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y$, and both $\cosh y$ & $\sinh y$ grow large as $y \to \infty$.

b) Note that $\cos z = \text{Re}(e^{iz})$. Thus $\text{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+100}\,dx\right) = \int_{-\infty}^{\infty} \frac{\cos x}{100+x^2}\,dx$ as desired.

Also, $|e^{i(x+iy)}| = |e^{-y+ix}| = e^{-y}$, which is bounded by 1 for $y>0$ (on the upper half plane). Thus, ~~the~~ we can use the Residue Thm in this situation

The residues ~~are~~ of $\frac{e^{iz}}{z^2+100}$ are $z^2+100=0 \to z = \pm 10i$, of which only $z = 10i$ lies above the real-axis.

$$\text{Res}(f; 10i) = \frac{e^{i(10i)}}{(z+10i)} = \frac{e^{-10}}{20i}$$

Thus, by the residue theorem, $\text{Re}\left(\int_{\gamma_R} \frac{e^{iz}}{z^2+100}\,dz\right) = \text{Re}\left(2\pi i\left(\frac{e^{-10}}{20i}\right)\right) = \text{Re}\left(\frac{\pi}{10e^{10}}\right) = \frac{\pi}{10e^{10}}$ for all R. Then, as $R \to \infty$, we find that

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos x}{100+x^2}\,dx = \frac{\pi}{10e^{10}}}$$