Final Exam for Math 403(02) (Spring 2020, Rutgers University, Dr. Z.) VERSION 1

Please Email DrZcomplex@gmail.com by May 8, 2020, 11:10am
Email with (Subject: Final) a .txt file attachment

FinalFirstNameLastName.txt

OR (if you don't have a text editor like notepad) a simple email message with the answers and explanations in plain English and in addition a .pdf file

FinalFirstNameLastName.pdf

(with a scan of the hand-written solutions) OR

FinalFirstNameLastNamePage1.jpg, FinalFirstNameLastNamePage2.jpg, ...

Open book (only the class textbook), and open notes (my notes). Any other help (from people or the internet) will be cheating.

The last twenty minutes should be for checking whether any of the answers are Nonsense. Add a sentence

I CHECKED FOR NONSENSE, and could not find any.

Every question is worth 15 points except one that is worth 20 points

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1. (15 points altogether) Let R be the following region in the complex plane

$$R = \{z : 5 \leq |z| < 7\} \quad .$$

(a) (2 points) Is it open? EXPLAIN
(b) (2 points) Is it closed? EXPLAIN
(c) (2 points) Is it Connected? EXPLAIN.
(d) (2 points) Is it Simply Connected? EXPLAIN
(e) (2 points) What is its interior?
(f) (2 points) What is its closure?
(g) (3 points) What is its boundary?

2. (15 points altogether) Recall that if $z = x + iy$ then $\bar{z} = x - iy$.
Decide whether $f(z) = (\bar{z})^2$ an analytic function in two different ways.

(a) (8 points) Using the definition of an analytic function in a domain as one for which

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \quad ,$$

exists in the sense of complex variable, i.e. that you always get the same answer no-matter which direction h approaches 0 on the complex plane.

(b) (7 points) Using the Cauchy-Riemann Equations.

3. (15 points) Find the root(s) of the quadratic equation

$$z^2 - (2+2i)z + 2i \;=\; 0 \quad .$$

4. (15 points) Using the well-known fact that the only zeros of $\sin z$ are on the real axis and are $n\pi$ (n integer) and the only zeros of $\cos z$ are on the real axis and are $(n+\frac{1}{2})\pi$ (n integer), find the change of argument of $f(z) = \cot\, z$ as it transverses the circle $|z| = 7$ in the **clockwise** direction. (Reminder: $\cot z = \frac{\cos z}{\sin z}$).

5. (15 points altogether) Find

(a) (12 points) The **maximum value** of $|z-i|^3$, in the closed disc $\{z : |z| \leq 3\}$. and the **location** (or locations) where it takes place. Explain!

(b) (3 points) The **minium value** of $|z-i|^3$, in the closed disc $\{z : |z| \leq 3\}$. and the **location** (or locations) where it takes place.Explain!

6. (15 points altogether) According to Cauchy's famous theorem, if $f(z)$ is analytic in a domain and Γ is a closed curve in that domain, then $\int_\Gamma f(z)\,dz = 0$.

Verify Cauchy's theorem directly for the special case where

- $f(z) = z$
- Γ is the boundary of triangle whose vertices are $z=0$, $z=1$ and $z=i$.

(a) (3 points) Compute $\int_{0\to 1} \; z\,dz \quad .$

(b) (8 points) Compute $\int_{1\to i} \; z\,dz \quad .$

(c) (3 points) Compute $\int_{i\to 0} \; z\,dz \quad .$

(d) (1 point) Then add them up and make sure that you indeed get 0. Show your work!

7. Compute the contour integral

$$\int_{|z|=5} \frac{(z-1)(z-2)}{(z-1-i)(z-4i)}\,dz \quad .$$

8. (15 points altogether) Use the Mean Value Theorem to compute the following integrals or explain why it is not applicable
(a) (8 points)

$$\int_0^{2\pi} \cos(\frac{\pi}{3} + 3e^{it})\,dt \quad .$$

(b) (7 points)

$$\int_0^{2\pi} \tan(\frac{\pi}{4} + 3e^{it})\, dt \quad .$$

9. (15 points altogether) (a) (5 points) Show that the following infinite series is convergent and (b) (10 points) determine its exact value

$$\sum_{n=0}^{\infty} \frac{(i\pi/2)^n}{n!} \quad .$$

10.(15 points altogether) Consider the function $f(z) = z^2$ defined on the disc $\{z \, : \, |z| < 1\}$.

(a) (5 points) What is its range, $f(D)$?

(b) (10 points) Is it one-to-one? If yes, prove it. If no, find two specific numbers that get mapped to the same point.

11. (15 points) Use the Taylor polynomial of degree 2 at $z_0 = 2i$ to approximate $f(z) = z^3$ at $z = 2i + \frac{1}{10}$.

12. (15 points) Find the zero (or zeros) of $f(z) = z^3 - 1 - i$ that lie in the first quadrant $\{z \, : \, Re\, z > 0 \quad , \quad Im\, z > 0\}$.

13. (20 points altogether) We need to compute the real integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{100 + x^2}\, dx$$

using residues, by considering a related contour integral, over a contour, let's call it γ_R, that consists of the circumference of the semi-circle $|z| = R$ **above** the x-axis, together with its base $\{z \, : \, z = x + i \cdot 0 \, , \, -R \leq x \leq R\}$, for very large R, and at the end take the limit $R \to \infty$.

(a) (4 points) Explain why considering the following natural contour integral

$$\int_{\gamma_R} \frac{\cos z}{100 + z^2}\, dz$$

does **not** help us in finding the value of $\int_{-\infty}^{\infty} \frac{\cos x}{100+x^2}\, dx$.

(b) (16 points) Find another integrand for which using residues **will** work, and use it to find the value of $\int_{-\infty}^{\infty} \frac{\cos x}{100+x^2}\, dx$.