Solutions to Real Quiz 8 of Dr. Z.'s Dynamical Models in Biology class

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1. Recall that the Hardy-Weinberg rule (where the fraction of the population that have genotype AA is u, the fraction that have genotype Aa is v (and hence the fraction of the population that has genotype aa is $w = 1 - u - v$) is:

$$(u, v) \to \left(u^2 + vu + \frac{1}{4}v^2, -2vu - 2u^2 + 2u - \frac{1}{2}v^2 + v\right)$$

(i) (5 points) If right now, $\frac{1}{2}$ of the population have genotupe AA, $\frac{1}{3}$ of the polpulation have genotype Aa, what is the fraction of aa genotypes **(ia)** Right now? **(ib)** In ten generations?

(ii) (5 points) Prove that the above equilibrium (after ten generations) is semi-stable.

Sol. of 1.(i)a: $w = 1 - u - v = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. So right now $\frac{1}{6}$ of the population has genotype aa.

Sol. of 1.(ib):

$$(\frac{1}{2}, \frac{1}{3}) \to \left((\frac{1}{2})^2 + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4}(\frac{1}{3})^2, -2\frac{1}{3}\frac{1}{2} - 2(\frac{1}{2})^2 + 2u - \frac{1}{2}(\frac{1}{3})^2 + \frac{1}{3}\right) = (\frac{4}{9}, \frac{4}{9})$$

So after one generation it is $(\frac{4}{9}, \frac{4}{9})$. But this now stays for ever after, in particular after ten generations. So $w = 1 - \frac{4}{9} - \frac{4}{9} = \frac{1}{9}$.

Ans. to 1.(ib) After ten generations $\frac{1}{9}$ of the population would have genotype aa.

So. to 1(ii) The Jacobian is:

$$JAC(u, v) \begin{bmatrix} 2u + v & u + v/2 \\ -2v - 4u + 2 & -2u - v + 1 \end{bmatrix}$$

Plugging in $u = \frac{4}{9}$ and $v = \frac{4}{9}$ we have

$$JAC(u, v) \begin{bmatrix} 2 \cdot \frac{4}{9} + \frac{4}{9} & \frac{4}{9} + (\frac{4}{9})/2 \\ -2\frac{4}{9} - 4\frac{4}{9} + 2 & -2\frac{4}{9} - \frac{4}{9} + 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \quad .$$

The **characteristic equation** is

$$(\frac{4}{3} - \lambda)(-\frac{1}{3} - \lambda) - (\frac{2}{3})(-\frac{2}{3}) = 0$$

Simplifying

$$\lambda^2 - \lambda = 0 \quad .$$

Factorizing:

$$(\lambda - 1)\lambda = 0 \quad .$$

So the eigenvalues are 1 and 0. Since the absolute values are ≤ 1 but one of them has absolute value 1, $(\frac{4}{9}, \frac{4}{9})$ it is **semi-stable**.