Solutions to Real Quiz 4 of Dr. Z.'s Dynamical Models in Biology class

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1.: Consider the non-linear recurrence

$$a(n+1) = \frac{1}{100}a(n)^4 - \frac{7}{100}a(n)^2 + \frac{53}{50}a(n) \quad .$$

(i) (3 points) Show that $x=-3$, $x=0$, $x=1$, $x=2$ are steady-states. Explain!

Sol. to 1(i): This is a first-order non-linear recurrence of the form

$$a(n+1) = f(a(n)) \quad ,$$

where the function $f(x)$ is

$$f(x) = \frac{1}{100}x^4 - \frac{7}{100}x^2 + \frac{53}{50}x \quad .$$

We have

$$f(-3) = \frac{1}{100}\cdot(-3)^4 - \frac{7}{100}\cdot(-3)^2 + \frac{53}{50}\cdot(-3) = -3$$

$$f(0) = \frac{1}{100}0^4 - \frac{7}{100}0^2 + \frac{53}{50}\cdot 0 = 0 \quad .$$

$$f(1) = \frac{1}{100}\cdot(1)^4 - \frac{7}{100}\cdot(1)^2 + \frac{53}{50}\cdot(1) = 1$$

$$f(2) = \frac{1}{100}\cdot(2)^4 - \frac{7}{100}\cdot(2)^2 + \frac{53}{50}\cdot(2) = 2$$

Hence indeed $x=-3$, $x=0$, $x=1$, $x=2$ are all steady-states, since they are fixed points of the underlying function $f(x)$, if the dynamical system happens to be at one of them, it stays there for ever.

(ii) (7 points) Which of these steady-states are stable? Explain!

Sol. of 1(ii):

We need to find $f'(x)$.

$$f'(x) = \frac{1}{25}x^3 - \frac{7}{50}x + \frac{53}{50} \quad .$$

Regarding $x=-3$ we have

$$f'(-3) = \frac{1}{25}(-3)^3 - \frac{7}{50}\cdot(-3) + \frac{53}{50} = \frac{2}{5} \quad .$$

Since the abslute value of $\frac{2}{5}$ (that happens to be $\frac{2}{5}$, since it is positive) is **less** than 1 it is a **stable** fixed point.

Regarding $x = 0$ we have

$$f'(0) = \frac{1}{25}(0)^3 - \frac{7}{50} \cdot (0) + \frac{53}{50} = \frac{53}{50} \quad .$$

Since the abslute value of $\frac{53}{50}$ (that happens to be $\frac{53}{50}$, since it is positive) is **more** than 1 it is an **unstable** fixed point.

Regarding $x = 1$ we have

$$f'(1) = \frac{1}{25}(1)^3 - \frac{7}{50} \cdot (1) + \frac{53}{50} = \frac{24}{25} \quad .$$

Since the abslute value of $\frac{24}{25}$ (that happens to be $\frac{24}{25}$, since it is positive) is **less** than 1 it is a **stable** fixed point.

Regarding $x = 2$ we have

$$f'(3) = \frac{1}{25}(2)^3 - \frac{7}{50} \cdot 2 + \frac{53}{50} = \frac{11}{10} \quad .$$

Since the abslute value of $\frac{11}{10}$ (that happens to be $\frac{11}{10}$, since it is positive) is **more** than 1 it is an **unstable** fixed point.