Solutions to Attendance Quiz for Lecture 7 of Dr. Z.'s Dynamical Models in Biology class

Name: Dr. Z.

1. Find all the steady-states of the non-linear recurrence

$$x(n+1) \;=\; \frac{5}{2}\,x(n)\,(1-x(n)) \quad .$$

Which of them is a stable steady-state?

What is $\lim_{n\to\infty} x(n)$ if $x(0)$ is between 0 and 1?

Sol. of 1:

The underlying function is

$$f(x) = \frac{5}{2}\,x\,(1-x) \quad .$$

We have to solve the algebraic equation $x = f(x)$, i.e.

$$x = \frac{5}{2}\,x\,(1-x) \quad .$$

Moving everything to the left we have

$$x - \frac{5}{2}\,x\,(1-x) = 0 \quad .$$

Factorizing

$$x(1 - \frac{5}{2}(1-x)) = 0 \quad .$$

Simplifying

$$x(1 - \frac{5}{2} + \frac{5}{2}x) = 0 \quad .$$

Simplifying more

$$x(-\frac{3}{2} + \frac{5}{2}x) = 0 \quad .$$

There are two solutions $x = 0$ and $x = (\frac{3}{2})/(\frac{5}{2}) = \frac{3}{5} \quad .$

Ans. to first part: The steady-states of this recurrence are $x = 0$ and $x = \frac{3}{5}$.

To decide about the stability we must first compute $f'(x)$.

Since $f(x) = \frac{5}{2}\,x - \frac{5}{2}\,x^2$, we have

$$f'(x) = \frac{5}{2} - 5\,x \quad .$$

When $x = 0$ we have

$$f'(0) = \frac{5}{2} \quad .$$

Since this is larger than 1 (in absolute value, but now it is positive so it is the same thing) $x = 0$ is **unstable**.

When $x = \frac{3}{5}$,

$$f'(x) = \frac{5}{2} - 5 \cdot \frac{3}{5} = -\frac{1}{2} \quad .$$

Since the absolute value of $-\frac{1}{2}$ (that is equal to $\frac{1}{2}$) is less than 1 this is **stable**.

Ans. to Second part

$x = 0$ is an unstable steady-state and $x = \frac{3}{5}$ is a stable steady-state.

Since there is only one stable steady-state if you start anywhere between 0 and 1 all the orbits converge to $\frac{3}{5}$.

Ans. to third part: For whatever initial value, $x(0)$ between 0 and 1,

$$\lim_{n \to \infty} x(n) = \frac{3}{5} \quad .$$