Solution to Attendance Quiz for Lecture 3 of Dr. Z.'s Dynamical Models in Biology class

1. Solve the following recurrence with the given initial conditions

$$a(n) \;=\; 5a(n-1) - 6a(n-2) \;+\; 2\,n - 7$$

subject to the initial conditions

$$a(0) = 2 \quad , \quad a(1) = 6 \quad .$$

Sol. to 1: **First** we use algebra to write the recurrence as

$$a(n) - 5a(n-1) + 6a(n-2) = 2n - 7 \quad .$$

where $a(n), a(n-1), \ldots$ are on the left side and everything else (an expression in n) is on the right side.

Step 1: Find the general solution of the **homogeneous** version

$$a(n) - 5a(n-1) + 6a(n-2) = 0 \quad ,$$

where the right side is replaced by 0.

The **characteristic equation** is

$$z^2 - 5z + 6 = 0 \quad .$$

Factoring

$$(z-2)(z-3) = 0 \quad .$$

So the **general solution of the homogeneous version is**

$$a(n) = C_1 2^n + C_2 3^n \quad ,$$

for some arbitrary constants.

Step 2: Find a *particular solution* of the original recurrence, by setting-up an appropriate **template**

• If the right side is a constant, then the template is $a(n) = c_0$

• If the right side is a polynomial of degree 1, then the template is $a(n) = c_0 + c_1 n$

In general, if the right side is a polynomial of degree k, then the the template is $a(n) = c_0 + c_1 n + \ldots + c_k n^k$.

More generally if the right side has the form $d^n Q(n)$ for *some* specific polynomial of degree k then the template is

$$a(n) = (c_0 + c_1 n + \ldots + c_k n^k) d^n \quad .$$

Going back to our problem the template is

$$a(n) = c_0 + c_1 n \quad .$$

Plugging it into the recurrence we get

$$a(n) - 5a(n-1) + 6a(n-2) = (c_0 + c_1 n) - 5(c_0 + c_1(n-1)) + 6(c_0 + c_1(n-2)) = (2c_0 - 7c_1) + 2c_1 n$$

Equating it to the right side gives:

$$(2c_0 - 7c_1) + 2\, c_1\, n \;=\; -7 \;+\; 2\, n \quad .$$

Equating the free term (coeff. of n^0) and the coefficients of n we get the system of two equations and two unknowns

$$2\, c_0 \;-7\, c_1 \;=\; -7 \quad , \quad 2\, c_1 \;=\; 2 \quad .$$

whose solution is $c_0 = 0$ and $c_1 = 1$. Going back to the template we have

a **particular solution** is $a(n) = n$

Step 3: The **general solution** of the original (inhomogeneous) recurrence is the general solution of the homogeneous version **plus** the above particular solution.

$$a(n) = C_1 2^n + C_2 3^n + n \quad .$$

Now, and only now, it is time to take advantage of the initial conditions $a(0) = 2 \quad , \quad a(1) = 6$

$$a(0) = C_1 2^0 + C_2 3^0 + 0 = C_1 + C_2 = 2 \quad .$$

$$a(1) = C_1 2^1 + C_2 3^1 + 1 = 2C_1 + 3C_2 + 1 = 6 \quad .$$

So we have to solve the system of two linear equations with two unknowns

$$C_1 + C_2 = 2 \quad , \quad 2C_1 + 3C_2 = 5 \quad ,$$

whose solution is $C_1 = 1$, $C_2 = 1$. Going back to the above general solution $a(n) = C_1 2^n + C_2 3^n + n$ we get

Ans. to 1: $a(n) = 2^n + 3^n + n$.