

Solutions to Attendance Quiz for Lecture 2 of Dr. Z.'s Dynamical Models in Biology class

1. Solve the following differential equation, subject to the given initial conditions

$$y''(x) - y(x) = 0 \quad ; \quad y(0) = 2 \quad , \quad y'(0) = 2 \quad .$$

Sol. to 1: The **characteristic equation** obtained by replacing $y''(x)$ by z^2 , $y'(x)$ by z , and $y(x)$ by 1 is

$$z^2 - 1 = 0 \quad .$$

Factoring we get

$$(z - 1)(z + 1) = 0 \quad ,$$

so there are two **distinct** real roots: $z_1 = -1$ and $z_2 = 1$.

The **general solution**, in general is $C_1 e^{z_1 x} + C_2 e^{z_2 x}$, so in this case it is

$$y(x) = C_1 e^x + C_2 e^{-x} \quad ,$$

where C_1, C_2 are *arbitrary constants* to be determined.

Now it is time to use the initial conditions. First we need to find the general expression for $y'(x)$.

$$y'(x) = C_1 e^x - C_2 e^{-x} \quad .$$

Plugging-in $x = 0$ (since $e^0 = 1$) we get

$$y(0) = C_1 + C_2$$

$$y'(0) = C_1 - C_2$$

So the system is

$$C_1 + C_2 = 2 \quad , \quad C_1 - C_2 = 2 \quad ,$$

whose solution is $C_1 = 2$, $C_2 = 0$.

Going back to the general solution we get

Ans. to 1: $y(x) = 2e^x + 0 \cdot e^{-x} = 2e^x$.

2. Find all the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{bmatrix} -2 & -3 \\ 4 & 5 \end{bmatrix}$$

Sol. to 2: The **characteristic matrix** is

$$\begin{bmatrix} -2-z & -3 \\ 4 & 5-z \end{bmatrix} ,$$

where z is a variable.

To find the **eigenvalues** you take the **determinant** and set it equal to 0.

$$(-2-z)(5-z) - (-3)(4) = 0 ,$$

that simplifies to

$$z^2 - 3z + 2 = 0$$

Factoring this is

$$(z-1)(z-2) = 0 .$$

Solving we get two **distinct** eigenvalues

$$z = 1 , \quad z = 2 .$$

We now need to find the corresponding **eigenvector** for each eigenvalue. Let's call it

$$\begin{bmatrix} a \\ b \end{bmatrix} .$$

For $z = 1$ the characteristic matrix is

$$\begin{bmatrix} -2-1 & -3 \\ 4 & 5-1 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 4 & 4 \end{bmatrix} =$$

Multiplying by the prospective eigenvector we get

$$\begin{bmatrix} -3 & -3 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

Spelling it out

$$-3a - 3b = 0 , \quad 4a + 4b = 0 .$$

These equations are redundant (as they should!) so you can take either of them and get

$$b = -a ,$$

where a is any non-zero number. Let's pick $a = 1$, so we have that an eigenvector corresponding to $z = 1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

For $z = 2$ the characteristic matrix is

$$\begin{bmatrix} -2-2 & -3 \\ 4 & 5-2 \end{bmatrix} = \begin{bmatrix} -4 & -3 \\ 4 & 3 \end{bmatrix} =$$

Multiplying by the prospective eigenvector we get

$$\begin{bmatrix} -4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .$$

Spelling it out

$$-4a - 3b = 0 \quad , \quad 4a + 3b = 0 \quad .$$

These equations are redundant (as they should!) so you can take either of them and get $b = -\frac{4}{3}a$. Since a can be any non-zero number, to make b nice let's take $a = 3$ and then $b = -4$ So we have that an eigenvector corresponding to $z = 2$ is $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

Ans. to 2: There are two eigenvalues $z = 1$ and $z = 2$

An eigenvector of $z = 1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

An eigenvector of $z = 2$ is $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$