Solutions to Attendance Quiz for Lecture 13 of Dr. Z.'s Dynamical Models in Biology class

1.: Find the steady-states and the stable steady-states of the 2-dimensional vector recurrence, or prove that they do not exist.

$$a_1(n+1) = \frac{a_1(n)}{a_2(n)}$$

$$a_2(n+1) = \frac{a_2(n)}{a_1(n)}$$

Sol. to 1:

The underlying transformation is

$$(x_1, x_2) \to (\frac{x_1}{x_2}, \frac{x_2}{x_1}) \quad .$$

i.e. $f_1(x_1, x_2) = \frac{x_1}{x_2}$, $f_2(x_1, x_2) = \frac{x_2}{x_1}$.

To get the **steady-states** you solve the **algebraic** system of two equations with two unknowns x_1, x_2.

$$x_1 = \frac{x_1}{x_2} \quad , \quad x_2 = \frac{x_2}{x_1} \quad .$$

Moving everything to the left

$$x_1 - \frac{x_1}{x_2} = 0 \quad , \quad x_2 - \frac{x_2}{x_1} = 0 \quad .$$

Factoring

$$x_1(1 - \frac{1}{x_2}) = 0 \quad , \quad x_2(1 - \frac{1}{x_1}) = 0 \quad .$$

From the first equation $x_1 = 0$ or $x_2 = 1$. From the second equation $x_2 = 0$ or $x_1 = 1$. But neither $x_1 = 0$ nor $x_2 = 0$ are legal since if you plug-them-in you get division by 0.

So Ans. to the first part: The only steady-state is $(1, 1)$.

Is it stable?

The **Jacobian matrix** is

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{x_2} & \frac{-x_2}{x_1^2} \\ \frac{-x_1}{x_1^2} & \frac{1}{x_1} \end{bmatrix}$$

This is a matrix of *functions*. Plugging in the candidate steady-state:

$$J(1,1) == \begin{bmatrix} \frac{1}{1} & \frac{-1}{1^2} \\ \frac{-1}{1^2} & \frac{1}{1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad .$$

We have to find the **eigenvalues**.

$$\det\left(\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 - (-1)^2 = 1 - 2\lambda + \lambda^2 - 1 = \lambda^2 - 2\lambda \quad .$$

So the characteristic equation is $\lambda^2 - 2\lambda = 0$. Factoring $\lambda(\lambda - 2) = 0$.

So the eigenvalues are 0 and 2. Since one of them is larger than 1 in absolute value this is **not** stable.

Ans. to 2nd part: There are no stable steady-states.