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MATH 336 (1), Dr. Z. , Exam 2, Wed., Nov. 26, 2025, 12:10-1:30pm, BECK 003

FRAME YOUR FINAL ANSWER(S) TO EACH PROBLEM

No Calculators! No books! No Notes! To ensure maximum credit, organize your work neatly and be sure to show all your work.

Do not write below this line

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1. 10 (out of 10)

2. 10 (out of 10)

3. 10 (out of 10)

4. 10 (out of 10)

5. 10 (out of 10)

6. 9 (out of 10)

7. 10 (out of 10)

8. 10 (out of 10)

9. 20 (out of 20)

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tot.: (out of 100)

99

1. (10 points) Solve explicitly the recurrence equation

$$x_n = x_{n-1} + 6x_{n-2} \quad ,$$

with initial conditions

$$x_0 = 2, x_1 = 1 \quad .$$

$r^2 = r + 6$ $\qquad$ $A + B = 2$

$r^2 - r - 6 = 0$ $\qquad$ $3A - 2B = 1$

$(r-3)(r+2) = 0$ $\qquad$ $3A - 2(2-A) = 1$

$r = 3 \quad r = -2$ $\qquad$ $5A - 4 = 1$

$A3^n + B(-2)^n = x_n$ $\qquad$ $A = 1$ $\qquad$ $3^n - 2^n = x_n$

Ans: $3^n + (-2)^n$ $\qquad$ $B = 1$

2. (10 points) In a certain species of animals, only one-year-old, two-year-old are fertile. The probabilities of a one-year-old, two-year-old, female to give birth to a new female are p_1, p_2, respectivley.
Assuming that there were c_0 females born at $n = 0$, c_1 females born at $n = 1$ Set up a recurrence that will enable you to find the **expected** number of females born at time n. In terms of c_0, c_1, p_1, p_2, how many females were born at $n = 4$?

$C_n = P_1 C_{n-1} + P_2 C_{n-2}$ $\qquad$ $C_0 = C_0 \quad C_1 = C_1$

$C_2 = P_1 C_1 + P_2 C_0$

$C_3 = P_1 C_2 + P_2 C_1 \to P_1(P_1 C_1 + P_2 C_0) + P_2 C_1$

$= (P_1^2 + P_2) C_1 + P_1 P_2 C_0$

$C_4 = P_1 C_3 + P_2 C_2$

$= P_1((P_1^2 + P_2) C_1 + P_1 P_2 C_0)$

$\boxed{C_4 = (P_1^3 + 2P_1 P_2) C_1 + (P_1^2 P_2 + P_2^2) C_0}$

3. (10 points) Prove that $a_1(n) = 2^{3^n}$ satisfies the non-linear recurrence equation

$$a(n) = a(n-1)^3 \ .$$

Is the following constant multiple of the seqeuncec $a_1(n)$, given by $a_2(n) = 2 \cdot 2^{3^n}$, also a solution? Why?

$$a_1(n-1)^3 = \left(2^{3^{n-1}}\right)^3 = 2^{3(3)^{n-1}} = 2^{3^n} = a_1(n)$$

$$a_2(n) = 2 \cdot 2^{3^n} = 2^{1+3^n}$$

$$a_2(n-1) = 2 \cdot 2^{3^n} = 2^{1+3^{n-1}}$$

$$a_2(n-1)^3 = \left(2^{1+3^{n-1}}\right)^3 = 2^{3(1+3^{n-1})} = 2^{3+3^n}$$

For a_2 to be a solution it would need $a_2(n) = a_2(n-1)^3$ which is impossible.

4. (10 points) Solve the following recurrence subject to the initla conditions.

$$a(n) = 3a(n-1) - 2a(n-2) + n \quad ; \quad a(0) = 2 \quad , \quad a(1) = 3 \ .$$

$$r^2 = 3r - 2 \to r^2 - 3r + 2 \to (r-1)(r-2)$$

$$r = 1, 2$$

$$a_h(n) = A(1)^n + B(2)^n \Rightarrow A + 2^n B \qquad An^2 + Bn + C$$

$$a_p(n-1) = An^2 + (-2A+B)n + (A-B+C)$$

$$a_p(n-2) = An^2 + (-4A+B)n + (4A-2B+C)$$

$$3a_p(n-1) - 2a_p(n-2) = An^2 + (2A+B)n + (-5A+B+C)$$

$$3a_p(n-1) - 2a_p(n-2) + n = An^2 + (2A+B+1)n + (-5A+B+C)$$

$$A = A$$

$$B = 2A + B + 1 \Rightarrow 0 = 2A + 1 \Rightarrow A = -\frac{1}{2}$$

$$C = -5A + B + C \Rightarrow 0 = -5A + B \Rightarrow B = 5A = -\frac{5}{2}$$

$$a_p = (n) = -\frac{1}{2}n^2 - \frac{5}{2}n = \frac{n^2+5n}{2} \qquad a(n) = A + B2^n - \frac{n^2+5n}{2}$$

$$a(1) = A + 2B - \frac{6}{2} = A$$

$$A + 2B = 6$$

$$(A+2B) - (A+B) = 6 - 2 = B = 4$$

$$a(n) = -2 + 4 \cdot 2^n - \frac{n^2+5n}{2}$$

5. (10 points) (a) Find all the steady-states of the non-linear recurrence

$$x(n) = kx(n-1)(1-x(n-1)) \quad ,$$

where k is a positive number between 0 and 4. Express your answer in terms of k (if needed).

(b) For which values of k is $x = 0$ the only stable steady-state? For which values of k is the other steady-state stable? For which values of k neither of them are?

a)

$$x = kx(1-x)$$

$$kx(1-x) - x = 0 \rightarrow x(k - kx - 1) = 0$$

$$x = 0 \text{ or } k - 1 - kx = 0 \Rightarrow kx = k-1 \Rightarrow x = 1 - \frac{1}{k}$$

$$x_1^n = 0 \quad x_2^n = 1 - \frac{1}{k}$$

b) $x_1 = 0$

$F(0) = k$ stable if $|k| < 1 \Rightarrow 0 < k < 1 \longrightarrow x=0$ is the only steady stable state

$x_2 = 1 - \frac{1}{k}$

$$F'(x_2) = k(1 - 2(1 - \tfrac{1}{k})) = k(1 - 2 + \tfrac{2}{k}) = -k + 2 = 2 - k$$

stable if $|2-k| < 1 \Rightarrow -1 < 2 - k < 1$

$\Rightarrow 3 < k < 1 \longrightarrow x = 1 - \frac{1}{k}$ is stable, $x=0$ is unstable

6. (10 points) Show that $x = 4$ is a steady-state of the discrete dynamical system, for any real number k,

$$x(n) = x(n-1) - \frac{(1 - x(n-1))\,(4 - x(n-1))}{k} \, ,$$

For what values of the parameter, k, is $x = 4$ a stable steady-state?

$$x(n) = 4 - \frac{(1-4)(4-4)}{k} = 4 \text{ for } k \neq 0$$

$$g(x) = x - \frac{(1-x)(4-x)}{k}$$

$$g'(x) = 1 - \frac{-5+2x}{k} \Rightarrow 1 + \frac{5-2x}{k}$$

$$g'(4) = 1 + \frac{5-8}{k} = 1 - \frac{3}{k} = \frac{k-3}{k}$$

Stable if $\left|\frac{k-3}{k}\right| < 1$

Spell out

$$-1 < 1 - \frac{3}{k} < 1$$

$$k > \frac{3}{2}$$

-1

7. (10 points) You enter a fair casino (where the prob. of winning a dollar is $\frac{1}{2}$ and the prob. of losing a dollar is $\frac{1}{2}$), and you must exit if you are broke or you have 200 dollars. If right now you have 150 dollars, how likely are you to exit a winner? How long would you expect to be in the casino until **you leave either a winner or loser?**

$P = \frac{i}{N}$ $i = 150$ $N = 200$

$P(\text{winner}) = \frac{150}{200} = 3/4$

b) $E_i = i(N-i)$

$E_{150} = 150(200-150) = 150(50)$
$= 7500$

8. (10 points) Find the equilibrium point(s) of the continuous dynamical system

$$x'(t) = 2x(t) - 3y(t) \quad , \quad y'(t) = 3x(t) - 2y(t)$$

Is it stable, semi-stable, or not stable? Explain!

$2x - 3y = 0$ $\qquad$ $3x - 2y = 0$ $\qquad$ $\frac{9}{2}y - 2y = \frac{5}{2}y = 0$ $\quad y = 0$

$x = \frac{3}{2}y$ $\qquad$ $3(\frac{3}{2}y) - 2y = 0$ $\qquad$ back in 1st eq

$x = 0$

only equilibrium is $(0,0)$

$A = \begin{pmatrix} 2-\lambda & -3 \\ 3 & -2-\lambda \end{pmatrix}$ $\quad (2-\lambda)(-2-\lambda) + 9$

$-4 + \lambda^2 + 9 = \lambda^2 + 5$

$\lambda^2 + 5 = 0 \Rightarrow \lambda = \pm i\sqrt{5}$

Trajectories are closed orbits around the origin the equilibrium is stable but not asymptotically stable

9. (20 points) (a) (10 points) Find all the steady-states of the discrete dynamical system

$$a_1(n+1) = \frac{a_1(n)}{2 - a_2(n)}$$

$$a_2(n+1) = \frac{a_2(n)}{2 - a_1(n)}$$

$a_1(n) = a_1(n+1) = x \qquad a_2(n) = a_2(n+1)$

$x = 0$

$x = \frac{x}{2-y} \qquad y = \frac{y}{2-x}$

$x = 0 \qquad y = \frac{y}{2}$

$y = 0$

$(0, 0)$

$x \neq 0$

$x = \frac{x}{2-y} \Rightarrow 2 - y \Rightarrow y = 1$

$y = \frac{y}{2-x} \Rightarrow 2 - x = 1 \Rightarrow x = 1$

$(1, 1)$

b (10 points) For these steady-states, which of them are stable? unstable? semi-stable? Explain.

$f_1 = \frac{a_1}{2-a_2} \qquad f_2 = \frac{a_2}{2-a_1}$

$\frac{\partial f_1}{\partial a_1} = \frac{1}{2-a_2} \qquad \frac{\partial f_1}{\partial a_2} = a_1 \frac{1}{(2-a_2)^2} \qquad \frac{\partial f_2}{\partial a_1} = a_2 \frac{1}{(2-a_1)^2} \qquad \frac{\partial f_2}{\partial a_2} = \frac{1}{2-a_1}$

$J(0,0) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1/2 \end{pmatrix} \qquad \lambda_1 = \frac{1}{2} \quad \lambda_2 = \frac{1}{2}$ both stable

$|\lambda| < 1$ stable

$J(1,1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = (\lambda - 1)(\lambda - 1) - 1$

$\lambda^2 - \lambda - \lambda + 1 - 1$

$\lambda^2 - 2\lambda$

$\lambda(\lambda - 2)$

$\lambda_1 = 0 \quad \lambda_2 = 2$

$|2| > 1$ unstable

$(0,0)$ stable
$(1,1)$ unstable