NAME: (print!) Daniyal Chaudhry

E-Mail address: dac433@scarletmail.rutgers.edu

MATH 336 (1), Dr. Z. , Final Exam, Tue., Dec.. 16, 2025, 8:00-11:00 am, BECK 003

FRAME YOUR FINAL ANSWER(S) TO EACH PROBLEM
No Calculators! No books! No Notes! To ensure maximum credit, organize your work neatly and be sure to show all your work.
Do not write below this line

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1. 15 (out of 15)

2. 15 (out of 15)

3. 15 (out of 15)

4. 15 (out of 15)

~~5. 14 (out of 15)~~

6. 14 (out of 15)

7. 15 (out of 15)

8. 15 (out of 15)

9. 15 (out of 15)

10. 15 (out of 15)

11. 15 (out of 15)

12. 15 (out of 15)

13. 20 (out of 20)

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tot.: (out of 200)

$\frac{184}{185} \cdot 200 =$ 199

1. (15 points) Solve explicitly the recurrence equation

$$x_n = 4x_{n-2} + 1 \quad ,$$

with initial conditions

$$x_0 = 1, x_1 = 2 \quad .$$

$X_n - 4X_{n-2} = 1$

① $x_n - 4x_{n-2} = 0 \rightarrow z^2 - 4 = 0 \rightarrow z = \pm 2 \rightarrow c_1(-2)^n + c_2(2)^n$

② undetermined α: $\alpha - 4\alpha = 1 \rightarrow -3\alpha = 1 \rightarrow \alpha = -1/3$

③ $x_n = c_1(-2)^n + c_2(2)^n - 1/3$ $\quad x_0 = 1 = c_1 + c_2 - 1/3 \quad x_1 = 2 = -2c_1 + 2c_2 - 1/3$

$2(c_1 + c_2 = 4/3)$
$+ \; -2c_1 + 2c_2 = 7/3$
$4c_2 = 15/3 \rightarrow c_2 = \frac{5}{4}$

$c_1 + \frac{15}{12} = \frac{16}{12}$
$c_1 = \frac{1}{12}$

$\boxed{X_n = \frac{1}{12}(-2)^n + \frac{5}{4}(2)^n - \frac{1}{3}}$

test: $X_0 = \frac{1}{12} + \frac{5}{4} - \frac{1}{3} = \frac{1}{12} + \frac{15}{12} - \frac{4}{12} = 1$ ✓
$X_1 = -\frac{2}{12} + \frac{30}{12} - \frac{4}{12} = \frac{24}{12}$ ✓ $\quad X_2 = \frac{4}{12} + \frac{60}{12} - \frac{4}{12} = 5$ ✓

2. (15 points) In a certain species of animals, only one-year-old, two-year-old are fertile. The probabilities of a one-year-old, two-year-old, female to give birth to a new female are $\frac{1}{2}, \frac{2}{3}$, respectivley.
Assuming that there were 10 females born at $n = 0$, 12 females born at $n = 1$. How many females were born at $n = 5$?

$f_1 = \frac{1}{2} \quad f_2 = \frac{2}{3}$ $\quad$ let c_n = # females born in year n

10 · 2yr olds $\frac{2}{3}$
12 1yr olds $\frac{1}{2}$

I feel like it was meant to be the other way around because that is much easier!

$c_n = f_1 c_{n-1} + f_2 c_{n-2}$

$c_n = \frac{1}{2} c_{n-1} + \frac{2}{3} c_{n-2}$

if $c_0 = 10, \; c_1 = 12$:

$c_2 = \frac{1}{2}(12) + \frac{2}{3}(10) = 6 + \frac{20}{3} = \frac{38}{3}$

$c_3 = \frac{1}{2}\left(\frac{38}{3}\right) + \frac{2}{3}(12) = \frac{19}{3} + \frac{24}{3} = \frac{43}{3}$

$c_4 = \frac{1}{2}\left(\frac{43}{3}\right) + \frac{2}{3}\left(\frac{38}{3}\right) = \frac{43}{6} + \frac{76}{9} = \frac{129}{18} + \frac{152}{18} = \frac{281}{18}$

$c_5 = \frac{1}{2}\left(\frac{281}{18}\right) + \frac{2}{3}\left(\frac{43}{3}\right) = \frac{281}{36} + \frac{86}{9} = \frac{281}{36} + \frac{344}{36}$

$\begin{array}{r} 281 \\ +\,344 \\ \hline 625 \end{array}$

$\boxed{\frac{625}{36}}$ females born at $n=5$

3. (15 points) Write the Maple command to solve the following recurrence (Do not solve it!)

$$a(n) = 10a(n-1) - 21a(n-2) + n^3 \quad , \quad a(0) = 5 \quad , \quad a(1) = 3$$

rsolve({a(n) = 10*a(n-1) - 21*a(n-2) + n^3, a(0)=5, a(1)=3}, a(n));

4. (15 points) Solve the following initial value problem

$$x''(t) + x(t) = 0 \quad , \quad x(0) = 1 \quad , \quad x'(0) = 0$$

$x'' + x = 0 \rightarrow r^2 + 1 = 0 \rightarrow r = \pm i$

solution form: $e^{(\alpha + \beta i)t} = \cancel{e^{\alpha t}}(A \sin(\beta t) + B \cos(\beta t))$

$x = A \sin t + B \cos t \rightarrow x' = A \cos t - B \sin t$

$x_0 = 1 = A(0) + B(1)$

$x'_0 = 0 = A(1) - B(0)$

$\left.\right\} \begin{matrix} B = 1 \\ A = 0 \end{matrix}$

$$\boxed{x(t) = \cos(t)}$$

(which makes sense, as $x'' = -\cos(t)$ and $\cos(0) = 1$, $\sin(0) = 0$!)

6. (15 points) Write the Maple command to solve the following intial value problem. Do not solve it

$$y^{(5)}(t)+y^{(3)}(t)+y(t)=\sin t \quad , \quad y^{(4)}(0)=1, y^{(3)}(0)=0, y^{(2)}(0)=-1, y'(0)=1, y(0)=3.$$

dsolve ({ diff (y(t), t $5) + diff (y(t), t$3) + y(t) = sin (t),
D(D(D(D(y))))(0) = 1, D(D(D(y)))(0) = 0, D(D(y))(0) = -1,
D(y)(0) = 1, y(0) = 3 }, y(t));

-1

7. (15 points) Solve the recurrence initial value problem

$$a(n)=a(n-5)-a(n-6) \quad ,$$

$$a(0)=0, a(1)=0, a(2)=0, a(3)=0, a(4)=0, a(5)=0 \quad .$$

Since all values presented are 0...

$\boxed{a(n) = 0}$

8. (15 points) (a) For which values of k is $x=1$ a stable steady-state of the ~~discrete~~ ~~continuous~~ dynamiical system

$$x(n)=x(n-1)+\frac{x(n-1)(1-x(n-1))(2-x(n-1))}{k}$$

$f(x) = x + \frac{x(1-x)(2-x)}{k} \longrightarrow f'(x) = 1 + \left(\frac{1}{k}\right)\left[\begin{matrix}(1-x)(2-x) + \\ (-1)(x)(2-x) + \\ (-1)(x)(1-x)\end{matrix}\right]$

such that the Jacobian $= [f'(x)]$

$J(1) = 1 + \frac{1}{k}\left[0 + (-1)(1)(2-1) + 0\right] = 1 + \frac{1}{k}(-1) = 1 - \frac{1}{k}$

which is stable when $\left|1 - \frac{1}{k}\right| < 1$ which holds for $\boxed{k > 0.5}$

(for all negative k, $1 - \frac{1}{k}$ is >1. for $0 \leq k \leq 0.5$, $1 - \frac{1}{k} \leq -1$)

9. (15 points) You enter a ~~fair~~ unfair casino (where the prob. of winning a dollar is 0.48 and the prob. of losing a dollar is 0.52), and you must exit if you are broke or you have 200 dollars. If right now you have 150 dollars, how likely are you to exit a winner?

$$\frac{1-\left(\frac{q}{p}\right)^n}{1-\left(\frac{q}{p}\right)^k}$$

$$\boxed{\frac{1-\left(\frac{0.52}{0.48}\right)^{150}}{1-\left(\frac{0.52}{0.48}\right)^{200}}}$$

solving this gives the likelihood of winning.

✓

10. (15 points) Find the equilibrium point(s) of the continuous dynamical system

$$x'(t) = x(t) - y(t) \quad , \quad y'(t) = -x(t) + y(t) \quad .$$

Is it stable, semi-stable, or not stable?. Explain!

$x' = x - y \quad y' = -x + y$: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

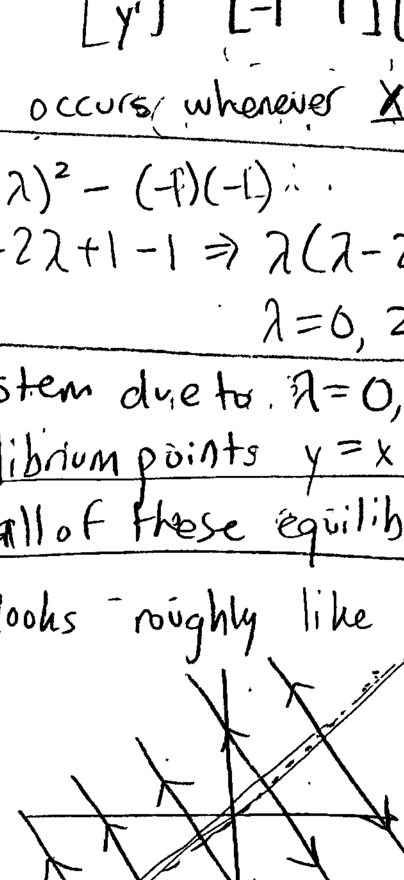
$x - y = 0$ and $-x + y = 0$ occurs whenever $x = y$.

$J = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow (1-\lambda)^2 - (-1)(-1)$

$\lambda^2 - 2\lambda + 1 - 1 \Rightarrow \lambda(\lambda - 2) = 0$

$\lambda = 0, 2$

This is a degenerate system due to $\lambda = 0$, which means there is a line of infinite equilibrium points $y = x$, i.e. all (x, x). The other eigenvalue $= 2$ is positive, so all of these equilibrium points are unstable. The phase plane looks roughly like



$y = x$ (eq. points)

(all solutions point away from eq. points)

✓

11. (15 points) Find all the steady-states of the discrete dynamical system

$$a_1(n+1) = \frac{a_1(n)}{2 - a_2(n)}$$

$$a_2(n+1) = \frac{a_2(n)}{2 - a_3(n)}$$

$$a_3(n+1) = \frac{a_3(n)}{4 - a_1(n)}$$

transformation

$(x_1, x_2, x_3) \longrightarrow \left(\frac{x_1}{2-x_2}, \frac{x_2}{2-x_3}, \frac{x_3}{4-x_1}\right)$

$X_1 = \frac{x_1}{2-x_2}$

$X_2 = \frac{x_2}{2-x_3}$ $\quad X_3 = \frac{x_3}{4-x_1}$

these can only be true if $x_1 = 3$, $x_2 = 1$, and $x_3 = 1$

OR: if $x_1 = 0$, $x_2 = 0$, $x_3 = 0$

the steady states are $(0, 0, 0)$ and $(3, 1, 1)$

12. (15 points) Find all the stable steady-states of the discrete dynamical system

$$a_1(n+1) = \frac{a_1(n)}{2 - a_2(n)}$$

$$a_2(n+1) = \frac{a_2(n)}{2 - a_3(n)}$$

$$a_3(n+1) = \frac{a_3(n)}{4 - a_1(n)}$$

The Jacobian $= \begin{bmatrix} \frac{1}{2-a_2} & \frac{a_1}{(2-a_2)^2} & 0 \\ 0 & \frac{1}{2-a_3} & \frac{a_2}{(2-a_3)^2} \\ \frac{a_3}{(4-a_1)^2} & 0 & \frac{1}{4-a_1} \end{bmatrix}$

$J(0,0,0) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

$\lambda\ 1/2, 1/2, 1/4$

as all $|\lambda| < 1$, $(0,0,0)$ is stable!

$J(3,1,1) = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$\rightarrow$ sum of λ = trace:

since the trace $= 3$, and there are 3λ, not all $|\lambda| < 1$!

so $(3,1,1)$ is semi or unstable, SO:

$\boxed{(0,0,0) \text{ is the only stable steady state}}$

13. (20 points) Recall that, in the Hardy-Weinberg model, if the fraction of the population of AA is u, the fraction of Aa is v (and hence the fraction of aa is $1-u-v$) then in the next generation it is

$$[u,v] \to [u^2 + v\,u + \frac{v^2}{4}, vu + 2u(1-u-v) + \frac{v^2}{2} + v\,(1-u-v)] .$$

If in this generation $\frac{1}{4}$ of the population is AA, $\frac{2}{5}$ and Aa. What fraction of the population will be aa in 100 generations?

NOW: $AA = \frac{1}{4}$, $Aa = \frac{2}{5}$, $aa = 0.35 = \frac{7}{20}$

0.25 $\quad$ 0.4

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futuregenerations:

$$u = \left(\tfrac{1}{4}\right)^2 + \left(\tfrac{2}{5}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{1}{4}\right)\left(\tfrac{2}{5}\right)^2 = \frac{1}{16} + \frac{2}{20} + \frac{4}{100}$$

$$= \frac{1}{16} + \frac{1}{10} + \frac{1}{25}$$

$$\frac{25}{400} + \frac{40}{400} + \frac{16}{400}$$

$$= \frac{81}{400}$$

$$v = \left(\tfrac{2}{5}\right)\left(\tfrac{1}{4}\right) + 2\left(\tfrac{1}{4}\right)\left(\tfrac{7}{20}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{2}{5}\right)^2 + \left(\tfrac{2}{5}\right)\left(\tfrac{7}{20}\right) =$$

$$\frac{2}{20} + \frac{14}{80} + \frac{4}{50} + \frac{14}{100}$$

$$= \frac{40}{400} + \frac{70}{400} + \frac{32}{400} + \frac{56}{400}$$

$$= \frac{198}{400}$$

$$w = 1 - \frac{81}{400} - \frac{198}{400} = \boxed{\frac{121}{400}}$$ in 100 gen will be aa

$$\begin{array}{r} 198 \\ +81 \\ \hline 279 \end{array} \qquad \begin{array}{r} 400 \\ -279 \\ \hline 121 \end{array}$$