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MATH 336 (1), Dr. Z. , Final Exam, Tue., Dec.. 16, 2025, 8:00-11:00 am, BECK 003

FRAME YOUR FINAL ANSWER(S) TO EACH PROBLEM
No Calculators! No books! No Notes! To ensure maximum credit, organize your work neatly and be sure to show all your work.
Do not write below this line

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1. (out of 15)
2. (out of 15)
3. (out of 15)
4. (out of 15)
5. (out of 15)
6. (out of 15)
7. (out of 15)
8. (out of 15)
9. (out of 15)
10. (out of 15)
11. (out of 15)
12. (out of 15)
13. (out of 20) – |

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tot.: (out of 200)

$\frac{184}{185} \cdot 200 = 199$

1. (15 points) Solve explicitly the recurrence equation

$$x_n = 4x_{n-2} + 1 \quad ,$$

with initial conditions

$$x_0 = 1, x_1 = 2 \quad .$$

$$\boxed{x_n = \frac{5}{4}(2)^n + \frac{1}{12}(-2)^n - \frac{1}{3}}$$

$\Rightarrow$ standard form: $x_{n+2} - 4x_n = 1$

1. Solve homogeneous version: $x_{n+2} - 4x_n = 0$

   $z^2 - 4 = 0$; $z = \pm 2$; general solution: $x_n = C_1 2^n + C_2(-2)^n$

2. Find particular solution: guess $x_n = \beta$

   $\beta - 4\beta = 1$; $\beta = -1/3$

   general solution: $x_n = C_1(2)^n + C_2(-2)^n - 1/3$

3. Use initial conditions: $x(0) = 1, x(1) = 2$

$$\begin{cases} C_1 + C_2 - 1/3 = 1 \ ; \ C_1 + C_2 = 4/3 \ ; \ C_2 = 4/3 - C_1 \\ 2C_1 - 2C_2 - 1/3 = 2 \ ; \ 2C_1 - 2C_2 = 7/3 \ ; \ 2C_1 - 2(4/3 - C_1) = 7/3 \ ; \ C_1 = \frac{5}{4} \end{cases}$$

$C_2 = \frac{1}{12}$

2. (15 points) In a certain species of animals, only one-year-old, two-year-old are fertile. The probabilities of a one-year-old, two-year-old, female to give birth to a new female are $\frac{1}{2}$, $\frac{2}{3}$, respectivley.
Assuming that there were 10 females born at $n = 0$, 12 females born at $n = 1$. How many females were born at $n = 5$?

$p_1 = \frac{1}{2}$; $p_2 = \frac{2}{3}$

$x_0 = 10$; $x_1 = 12$

general recursion: $x_n = p_2 x_{n-2} + p_1 x_{n-1}$

$x_2 = p_2 x_0 + p_1 x_1 = \left(\frac{2}{3} \cdot 10\right) + \left(\frac{1}{2} \cdot 12\right) = \frac{38}{3}$

$x_3 = p_2 x_1 + p_1 x_2 = \left(\frac{2}{3} \cdot 12\right) + \left(\frac{1}{2} \cdot \frac{38}{3}\right) = 8 + \frac{38}{6} = \frac{86}{6} = \frac{43}{3}$

$x_4 = p_2 x_2 + p_1 x_3 = \left(\frac{2}{3} \cdot \frac{38}{3}\right) + \left(\frac{1}{2} \cdot \frac{43}{3}\right) = \frac{281}{18}$

$x_5 = p_2 x_3 + p_1 x_4 = \left(\frac{2}{3} \cdot \frac{43}{3}\right) + \left(\frac{1}{2} \cdot \frac{281}{18}\right)$

$= \frac{86}{9} + \frac{281}{36} = \frac{625}{36}$

$\boxed{\text{at } n=5, \ \frac{625}{36} \text{ females born}}$

3. (15 points) Write the Maple command to solve the following recurrence (Do not solve it!)

$$a(n) = 10a(n-1) - 21a(n-2) + n^3 \quad , \quad a(0) = 5 \quad , \quad a(1) = 3$$

$\boxed{\texttt{S:= rsolve(\{a(n)=10*a(n-1)-21*a(n-2)+n^3, a(0)=5, a(1)=3\}, \{a(n)\});}}$

4. (15 points) Solve the following initial value problem

$$x''(t) + x(t) = 0 \quad , \quad x(0) = 1 \quad , \quad x'(0) = 0$$

$z^2+1=0$; $z=\pm i$

If $z=\lambda \pm \mu i$, general solution is...

$x(t) = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t)$

here, $\lambda = 0$, $\mu = 1$!

general solution: $x(t) = c_1 \cos(t) + c_2 \sin(t)$

$x'(t) = -c_1 \sin(t) + c_2 \cos(t)$

$\begin{cases} \text{From } x(0)=1,\ c_1\cos(0)+c_2\sin(0)=1;\ c_1=1 \\ \text{From } x'(0)=0,\ -c_1\sin(0)+c_2\cos(0)=0;\ c_2=0 \end{cases}$

$\boxed{x(t) = \cos(t)}$

6. (15 points) Write the Maple command to solve the following intial value problem. Do not solve it

$$y^{(5)}(t)+y^{(3)}(t)+y(t) = \sin t \quad , \quad y^{(4)}(0) = 1, y^{(3)}(0) = 0, y^{(2)}(0) = -1, y'(0) = 1, y(0) = 3.$$

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S:= dsolve({diff(y(t),t,t,t,t,t) + diff(y(t),t,t,t) + y(t) = sin(t), diff(y(t),0,0,0,0) = 1,
     diff(y(t),0,0,0) = 0, diff(y(t),0,0) = -1, diff(y(t),0) = 1, y(0) = 3},
    {y(t)}):

    y:= subs(S,y(t)):
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7. (15 points) Solve the recurrence initial value problem

$$a(n) = a(n-5) - a(n-6) \quad ,$$

$$a(0) = 0, a(1) = 0, a(2) = 0, a(3) = 0, a(4) = 0, a(5) = 0 \quad .$$

$\boxed{a(n) = 0}$

It recurrence has initial conditions all of
Zero, it is bound to stay there for all time

8. (15 points) (a) For which values of k is $x = 1$ a stable steady-state of the discrete ~~continuous~~ dynamiical system

$$x(n) = x(n-1) + \frac{x(n-1)(1-x(n-1))(2-x(n-1))}{k}$$

For a 1-dimensional,
1st order, nonlinear
discrete dynamical system,
a steady state
stable if $|f'(\bar{x}_i)| < 1$

$\Rightarrow$ standard form: $x(n+1) = x(n) + \dfrac{x(n)(1-x(n))(2-x(n))}{k}$

$f(x) = x + \dfrac{x(1-x)(2-x)}{k} = x + \dfrac{(x-x^2)(2-x)}{k} = x + \dfrac{2x - x^2 - 2x^2 + x^3}{k}$

$= x + \dfrac{x^3 - 3x^2 + 2x}{k}$; $f'(x) = 1 + \dfrac{3x^2 - 6x + 2}{k}$

$f'(1) = 1 + \dfrac{3-6+2}{k} = 1 - \dfrac{1}{k}$; $\left|1 - \frac{1}{k}\right| < 1$; $-1 < 1 - \frac{1}{k} < 1$

$-2 < -\frac{1}{k} < 0$

$2 > \frac{1}{k} > 0$; $k > \frac{1}{2}$

$\boxed{k > \frac{1}{2}}$

unfair

9. (15 points) You enter ~~a fair~~ casino (where the prob. of winning a dollar is 0.48 and the prob. of losing a dollar is 0.52), and you must exit if you are broke or you have 200 dollars. If right now you have 150 dollars, how likely are you to exit a winner?

$L=200$, $n=150$

$p=0.48$, $q=0.52$

$$\boxed{\mathbb{P}(\text{win}) = \frac{1-\left(\frac{0.52}{0.48}\right)^{150}}{1-\left(\frac{0.52}{0.48}\right)^{200}}}$$

10. (15 points) Find the equilibrium point(s) of the continuous dynamical system

$\mathbb{R}$ part of λ

$$x'(t) = x(t) - y(t) \quad , \quad y'(t) = -x(t) + y(t) \quad .$$

Is it stable, semi-stable, or not stable?. Explain!

$$\begin{cases} \frac{dx}{dt} = x-y \\ \frac{dy}{dt} = -x+y \end{cases} \quad ?$$

solve

$x-y=0$ and $-x+y=0$ ...

Equilibrium along $x=y$

$$J = \begin{bmatrix} \frac{\partial x}{\partial f_1} & \frac{\partial y}{\partial f_1} \\ \frac{\partial x}{\partial f_2} & \frac{\partial y}{\partial f_2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\det([J]) = (1-\lambda)(1-\lambda) - (-1)(-1)$

$= \cancel{1} - 2\lambda + \lambda^2 - \cancel{1}$

$\lambda^2 - 2\lambda = 0$, $\lambda(\lambda-2) = 0$

$\lambda_1 = 0$, $\lambda_2 = 2$

Semi-stable

Equilibrium: $x=y$
Semi-stable: one eigenvalue whose real part equals 0 and one eigenvalue whose real part is greater than 0.

11. (15 points) Find all the steady-states of the discrete dynamical system

$$a_1(n+1) = \frac{a_1(n)}{2 - a_2(n)}$$

$$a_2(n+1) = \frac{a_2(n)}{2 - a_3(n)}$$

$$a_3(n+1) = \frac{a_3(n)}{4 - a_1(n)}$$

$(x, y, z) \rightarrow \left(\frac{x}{2-y}, \frac{y}{2-z}, \frac{z}{4-x}\right)$

Potential SS's:
$\boxed{(0,0,0) \quad (3,1,1)}$ $(0,1,1)$ $(0,0,1)$
$(3,0,0)$ $(0,1,0)$
$(3,1,0)$
$(3,0,1)$

only three satisfy system of eqns.

$$\begin{cases} x = \frac{x}{2-y} \; ; \; x - \frac{x}{2-y} = 0 \; ; \; x\left(1 - \frac{1}{2-y}\right) = 0 \; ; \; x = 0 \text{ OR } y = 1 \\ y = \frac{y}{2-z} \; ; \; y - \frac{y}{2-z} = 0 \; ; \; y\left(1 - \frac{1}{2-z}\right) = 0 \; ; \; y = 0 \text{ OR } z = 1 \\ z = \frac{z}{4-x} \; ; \; z - \frac{z}{4-y} = 0 \; ; \; z\left(1 - \frac{1}{4-y}\right) = 0 \; ; \; z = 0 \text{ OR } x = 3 \end{cases}$$

$\boxed{\text{Steady States: } (0,0,0), (3,1,1)}$

12. (15 points) Find all the stable steady-states of the discrete dynamical system

$$a_1(n+1) = \frac{a_1(n)}{2 - a_2(n)}$$

$$a_2(n+1) = \frac{a_2(n)}{2 - a_3(n)}$$

$$a_3(n+1) = \frac{a_3(n)}{4 - a_1(n)}$$

$\boxed{\text{Stable Steady State: } (0,0,0)}$

For a K-dimensional, 1st order, discrete dynamical system, steady states are stable if the absolute value of all their eigenvalues is less than 1.

$$J = \begin{bmatrix} \frac{\partial x}{\partial f_1} & \frac{\partial y}{\partial f_1} & \frac{\partial z}{\partial f_1} \\ \frac{\partial x}{\partial f_2} & \frac{\partial y}{\partial f_2} & \frac{\partial z}{\partial f_2} \\ \frac{\partial x}{\partial f_3} & \frac{\partial y}{\partial f_3} & \frac{\partial z}{\partial f_3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2-y} & \frac{x}{(2-y)^2} & 0 \\ 0 & \frac{1}{2-z} & \frac{y}{(2-z)^2} \\ \frac{z}{(4-x)^2} & 0 & \frac{1}{4-x} \end{bmatrix}$$

$$J(0,0,0) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

For a matrix full of zeros everywhere except the main diagonal, eigenvalues are the entries along the main diagonal; here it's $\frac{1}{2}$ (multiplicity of 3)
$|\frac{1}{2}| < 1$, so stable

$$\begin{array}{l} J(3,1,1) \\ \quad x \; y \; z \end{array} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[\text{form}]{\text{row echelon}} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

On main diagonal, eigenvalues do not have all absolute values < 1, so not a stable steady state

$-\frac{5}{20} - \frac{8}{20}$ $\frac{4}{25} \cdot \frac{1}{4} = \frac{1}{25}$

$1 - \frac{1}{4} - \frac{2}{5}$ $1-\frac{13}{20} = \frac{7}{20}$

13. (20 points) Recall that, in the Hardy-Weinberg model, if the fraction of the population of AA is u, the fraction of Aa is v (and hence the fraction of aa is $1-u-v$) then in the next generation it is

$$[u,v] \to [u^2 + vu + \frac{v^2}{4}, vu + 2u(1-u-v) + \frac{v^2}{2} + v(1-u-v)] \quad .$$

If in this generation $\frac{1}{4}$ of the population is AA, $\frac{2}{5}$ and Aa. What fraction of the population will be aa in 100 generations?

$u = \frac{1}{4}, v = \frac{2}{5}$

eqn 1: $(\frac{1}{4})^2 + (\frac{1}{4} \cdot \frac{2}{5}) + \frac{(\frac{2}{5})^2}{4} = \frac{1}{16} + \frac{1}{10} + \frac{1}{25}$

eqn 2: $(\frac{1}{4} \cdot \frac{2}{5}) + 2(\frac{1}{4})(1 - \frac{1}{4} - \frac{2}{5}) + \frac{(\frac{2}{5})^2}{2} + \frac{2}{5}(1 - \frac{1}{4} - \frac{2}{5})$.

$= \frac{1}{10} + (\frac{1}{2} \cdot \frac{7}{20}) + \frac{2}{25} + (\frac{2}{5} \cdot \frac{7}{20}) = \frac{1}{10} + \frac{7}{40} + \frac{2}{25} + \frac{14}{100}$

fraction of aa (w) in population $= 1 - [(\frac{1}{16} + \frac{1}{10} + \frac{1}{25}) + (\frac{1}{10} + \frac{7}{40} + \frac{11}{50})]$

After the n+1 generation, population stabilizes and will exhibit these genetic ratios for the long-term

$\boxed{\text{In 100 years, } w = 1 - [(\frac{1}{16} + \frac{1}{10} + \frac{1}{25}) + (\frac{1}{10} + \frac{7}{40} + \frac{9}{50})]}$