SI and SIR Models

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Conjecture 1

CONJECTURE 1 *Prove if $R_0 = a/(b+c) > 1$, then the solution (x_n, y_n) to the first order system (1) and (2) satisfies*

$$\lim_{n\to\infty} (x_n, y_n) = (x^*, y^*),$$

where x^* and y^* are the positive solutions of (3).

$$x_{n+1} = x_n(1 - b - c) + y_n(1 - \exp(-ax_n)) \qquad (1)$$

$$y_{n+1} = (1 - y_n)b + y_n \exp(-ax_n), \qquad (2)$$

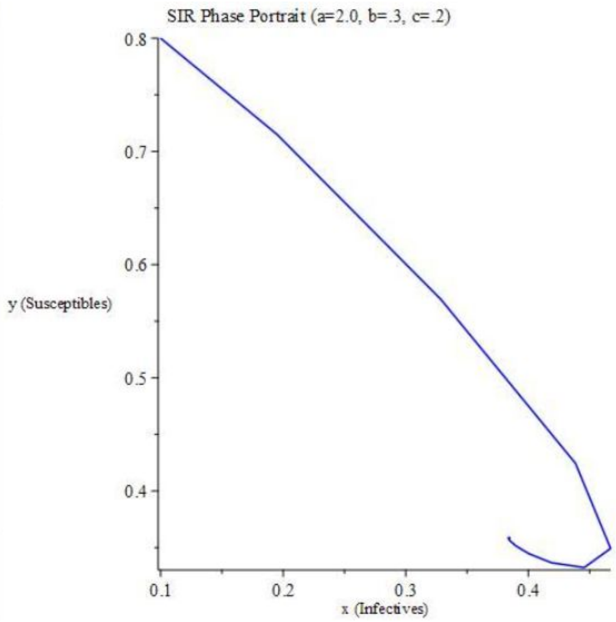
$$(b + c)x^* = y^*(1 - \exp(-ax^*)) \quad \text{and} \quad y^* = 1 - x^*(1 + c/b). \qquad (3)$$

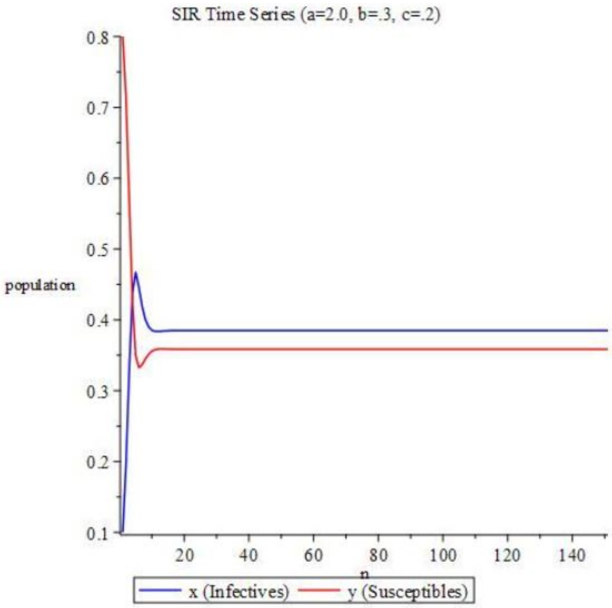
For the SIR model, if a/(b+c) > 1, then:

- Equilibrium exists
- Orbit converges to equilibrium
- Stable

```
# --- INITIAL CONDITIONS ---
x0_sir := 0.1 :   # initial infectives (SIR)
y0_sir := 0.8 :   # initial susceptibles (SIR)
x0_si := 0.1 :    # initial condition for SI stable
x1_si := 0.15 :   # second initial condition for SI (2nd order)
x0_uns := 0.5 :   # initial condition for SI unstable
```

Conjecture 1 Graphs

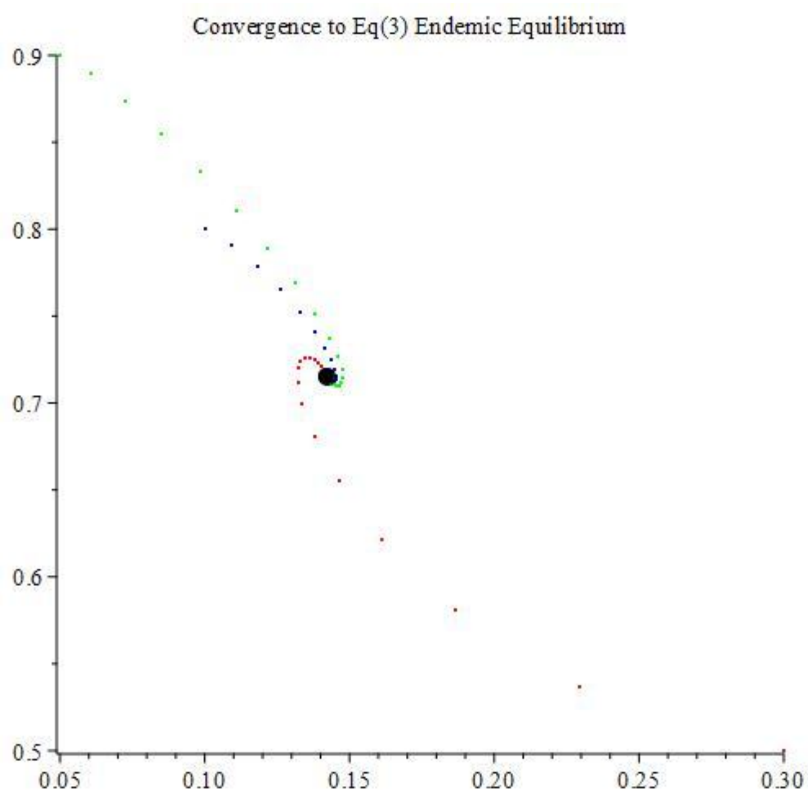




R = a/(b + c) = 4

- Endemic equilibrium is reached

Conjecture 1 Graphs



a = 0.8, b = 0.3, c = 0

Endemic equilibrium is reached for initial conditions:

- [0.1, 0.8]
- [0.3, 0.5]
- [0.05, 0.9]

```
T_SI := AllenSIR(0.8, 0.3, 0, x, y);
orb_SI := ORB(T_SI, [x, y], [0.1, 0.9], 0, 300)[-1];
```

$$T\_SI := [0.7x + y(1 - e^{-0.8x}), 0.3 - 0.3y + y e^{-0.8x}]$$

$$orb\_SI := [0.5384515242, 0.4615484758]$$

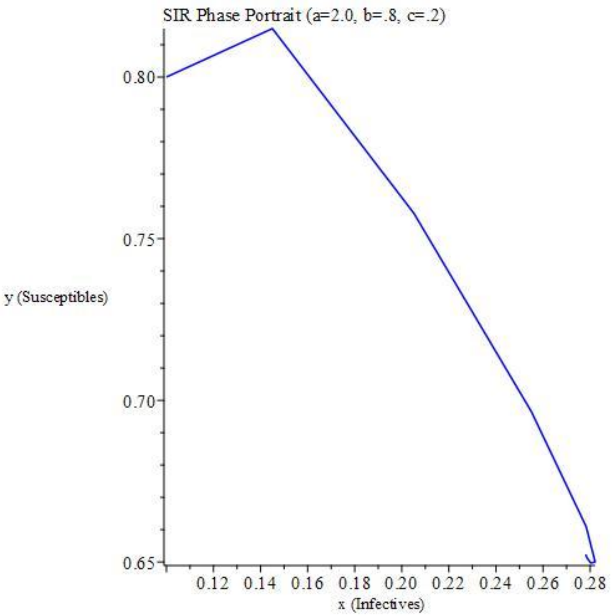
```
J := JAC(T, [x, y]);
J_endemic := subs({x = endemic_pt[1], y = endemic_pt[2]}, J);
Eigenvalues(Matrix(evalf(J_endemic)));
```

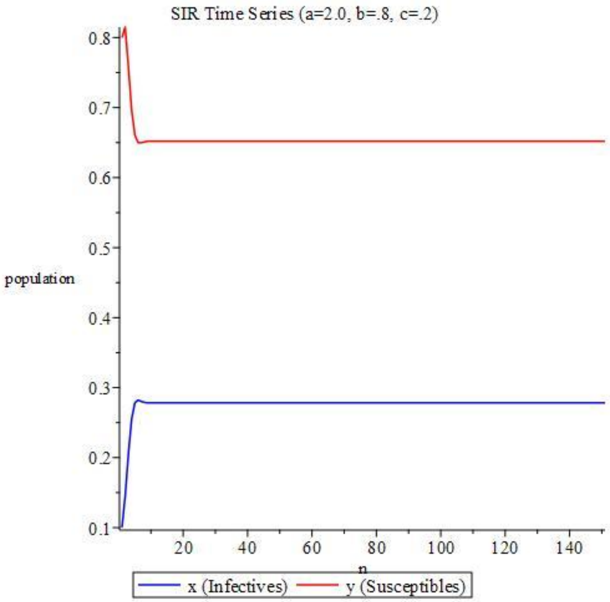
$$J := \left[\left[\frac{1}{3} + y e^{-x}, 1 - e^{-x}\right], \left[-y e^{-x}, -\frac{1}{3} + e^{-x}\right]\right]$$

$$J\_endemic := \left[\left[\frac{1}{3} + 0.7152506526\, e^{-0.1423746737}, 1 - e^{-0.1423746737}\right], \left[-0.7152506526\, e^{-0.1423746737}, -\frac{1}{3} + e^{-0.1423746737}\right]\right]$$

$$\begin{bmatrix} 0.743815222950000 + 0.195659617986750\, I \\ 0.743815222950000 - 0.195659617986750\, I \end{bmatrix}$$

Conjecture 1 Graphs

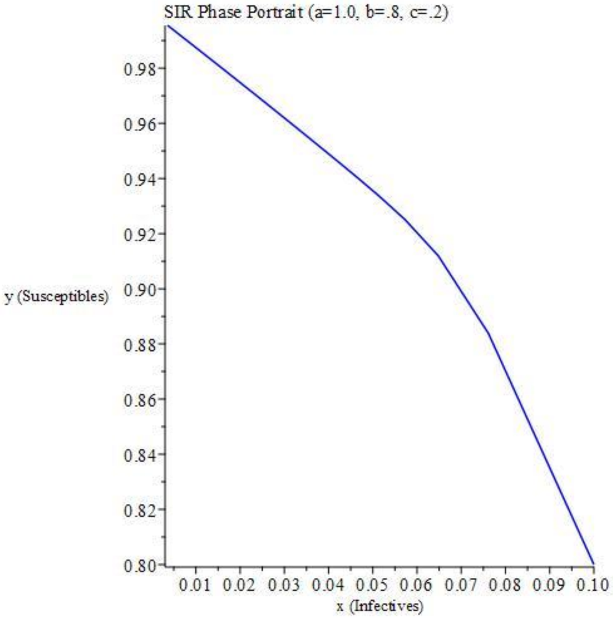


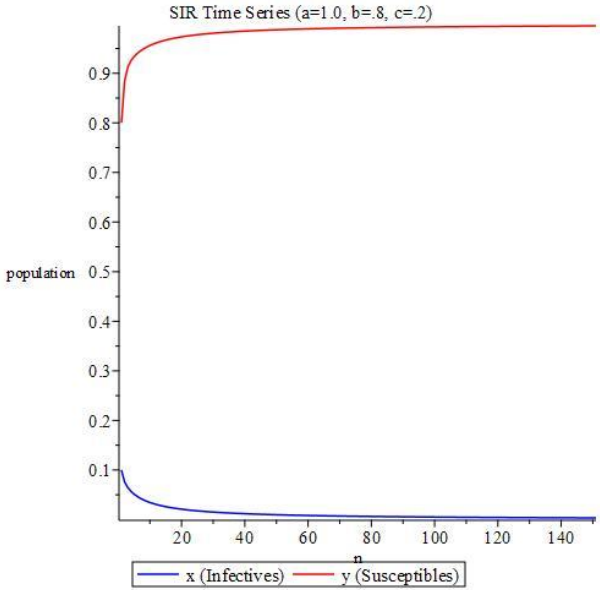


R = a/(b + c) = 2

- Endemic equilibrium is reached

Conjecture 1 Graphs

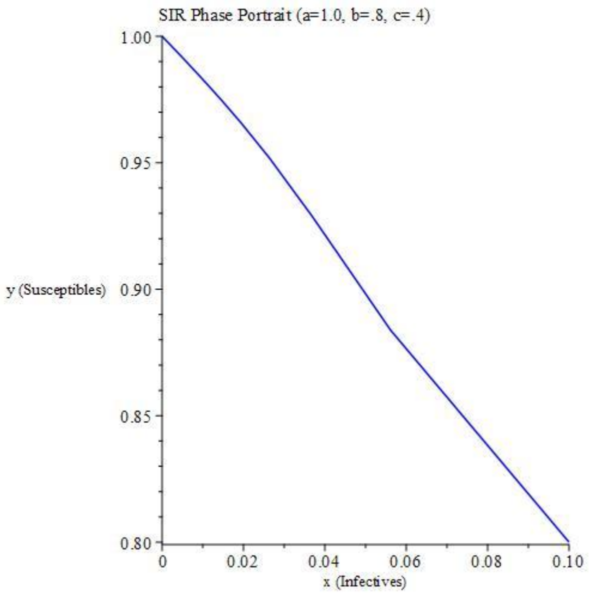


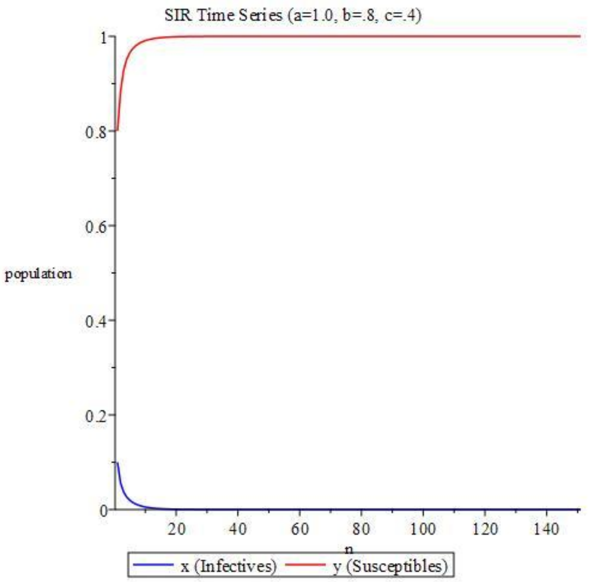


R = a/(b + c) = 1

- Virus dies out of population

Conjecture 1 Graphs





R = a/(b + c) = 0.833

- Virus dies out of population

Conjecture 2

CONJECTURE 2 *Prove if $R_0 = a/b > 1$, then the solution of (4) satisfies*

$$\lim_{n \to \infty} x_n = x^*,$$

where x^* is the positive solution of

$$bx^* = (1 - x^*)(1 - exp(-ax^*)).$$

$$x_{n+1} = x_n(1 - b) + (1 - x_n)(1 - \exp(-ax_{n-1})), \quad n = 1, 2, \ldots, \quad (4)$$

```
# --- INITIAL CONDITIONS ---
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x0_si := 0.1 :    # initial condition for SI stable
x1_si := 0.15 :    # second initial condition for SI (2nd order)
x0_uns := 0.5 :    # initial condition for SI unstable
```

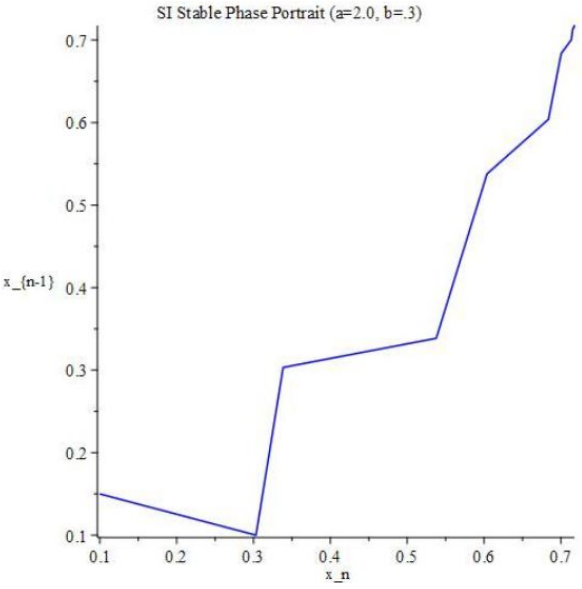
For SI model, if a/b > 1, then:

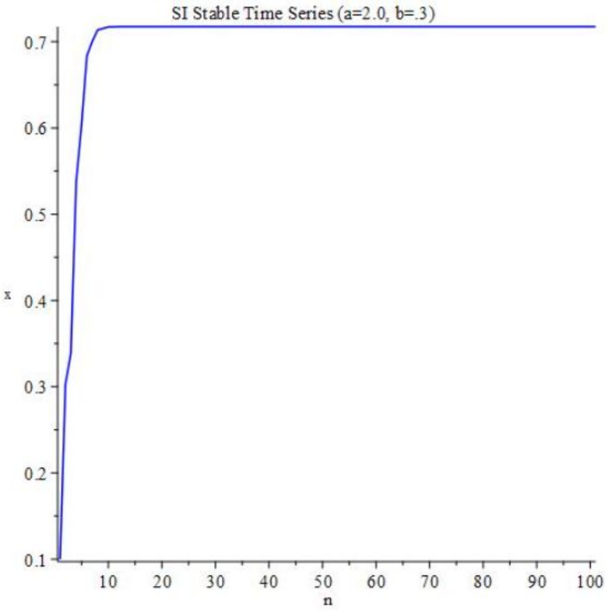
- Equilibrium exists
- Absolute value of eigenvalues are less than 1
- Orbits converge

If a/b < 1, then:

- Equilibrium becomes unstable
- Orbits oscillate/diverge

Conjecture 2 Graphs (Stable)

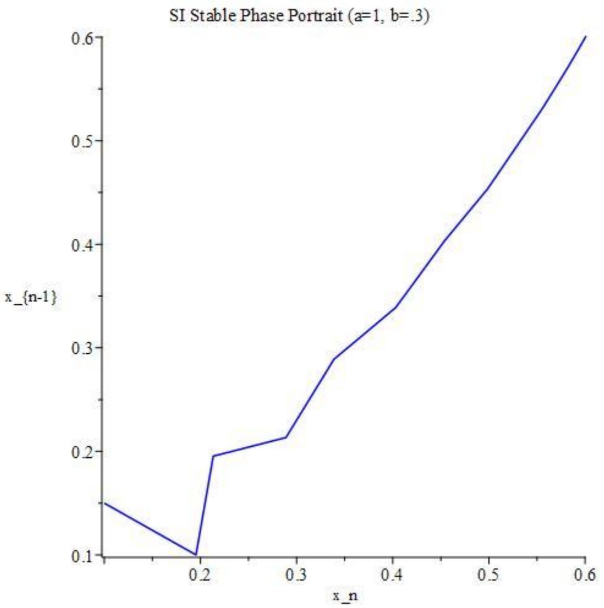


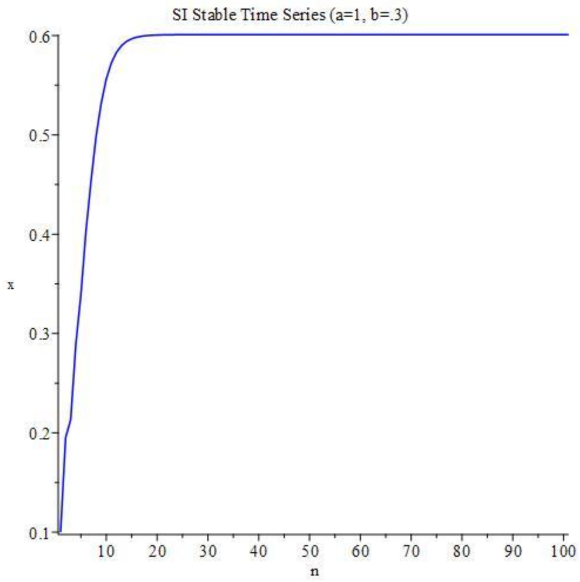


R = a/b = 6.667

- Endemic equilibrium is reached

Conjecture 2 Graphs (Stable)

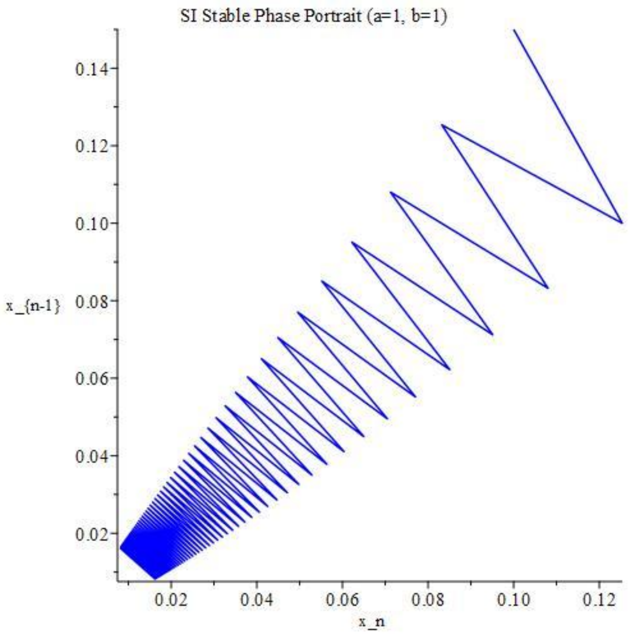


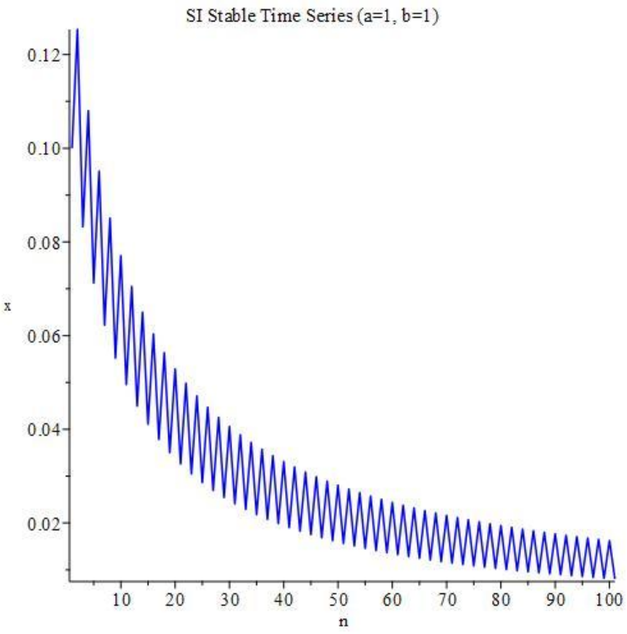


R = a/b = 3.3333

- Endemic equilibrium is reached

Conjecture 2 Graphs (Stable-ish)

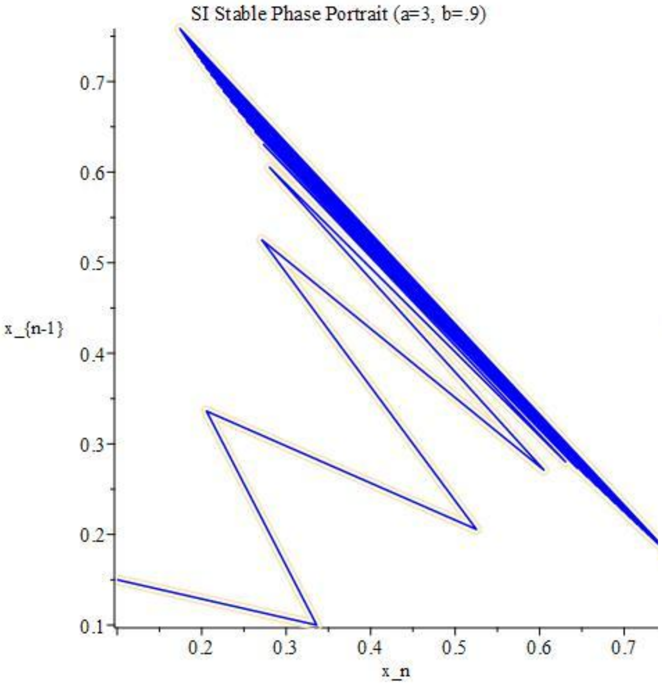


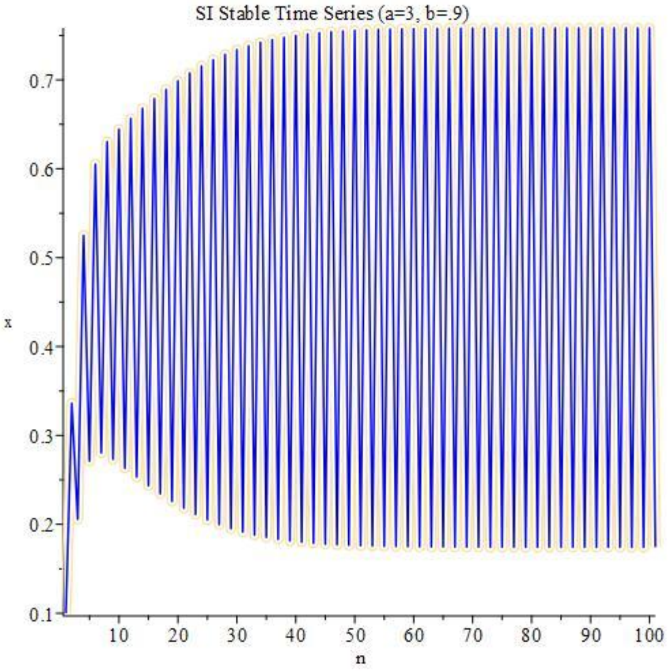


R = a/b = 1

- Oscillations occur

Conjecture 2 Graphs (Unstable)

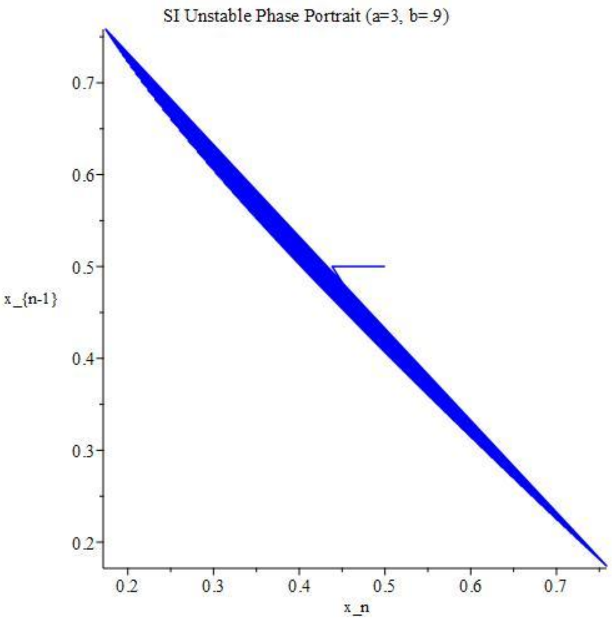


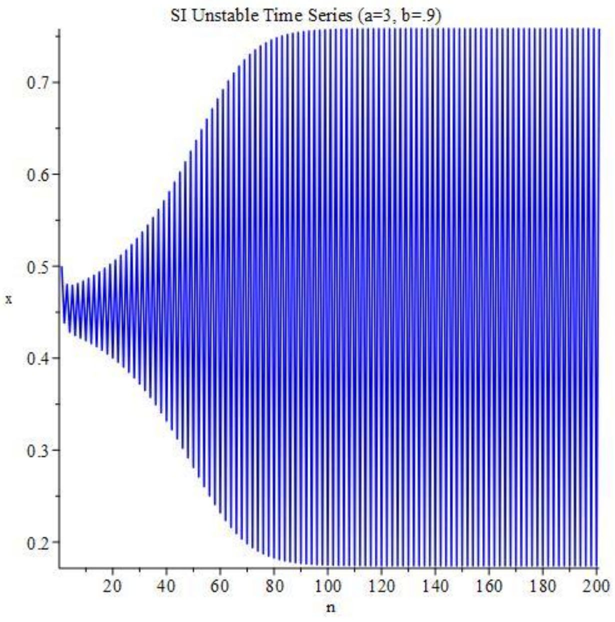


R = a/b = 3.333

- Oscillations occur

Conjecture 2 Graphs (Unstable)

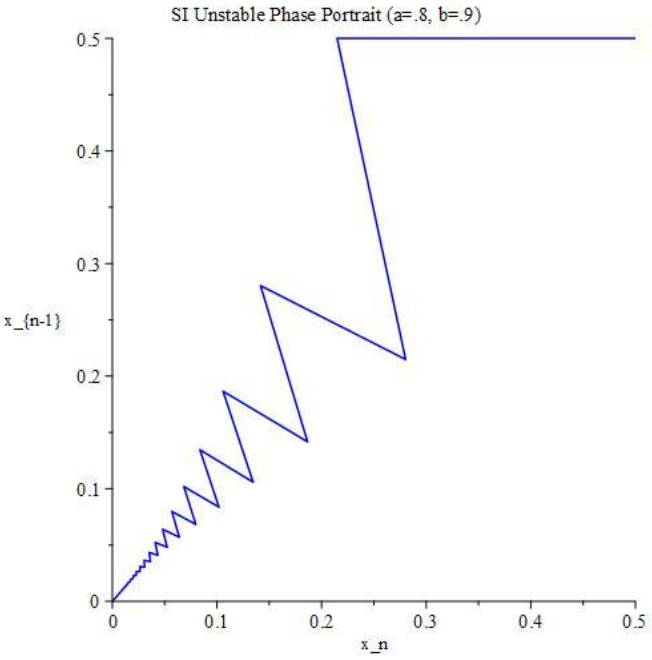


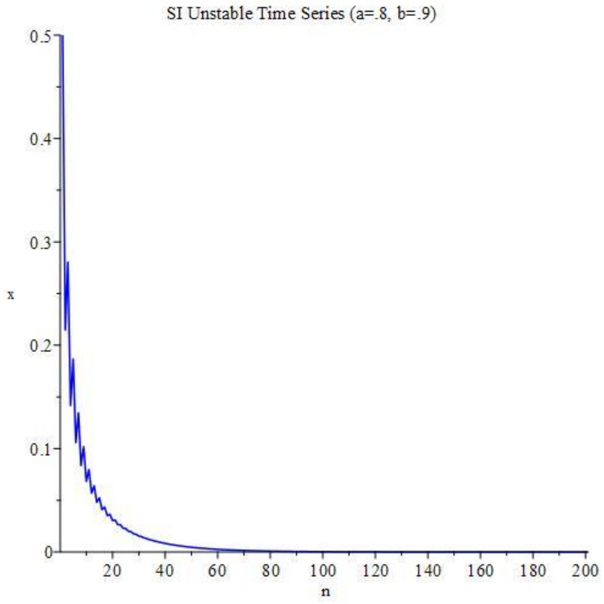


R = a/b = 3.333

- Oscillations occur

Conjecture 2 Graphs (Unstable)

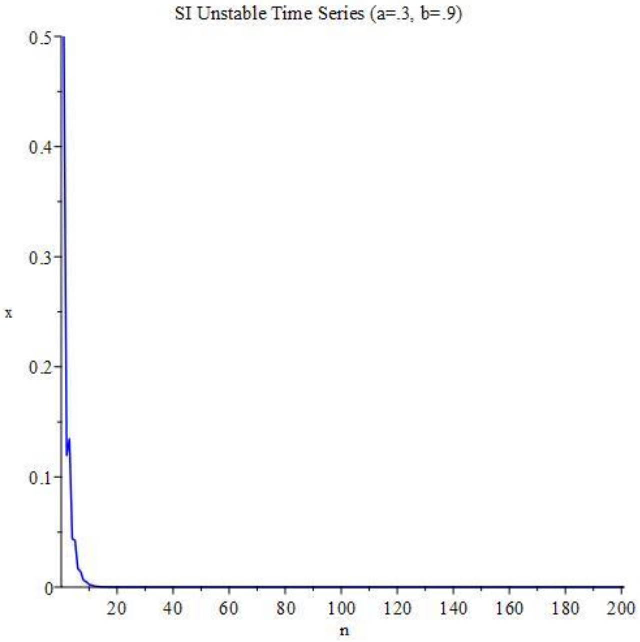


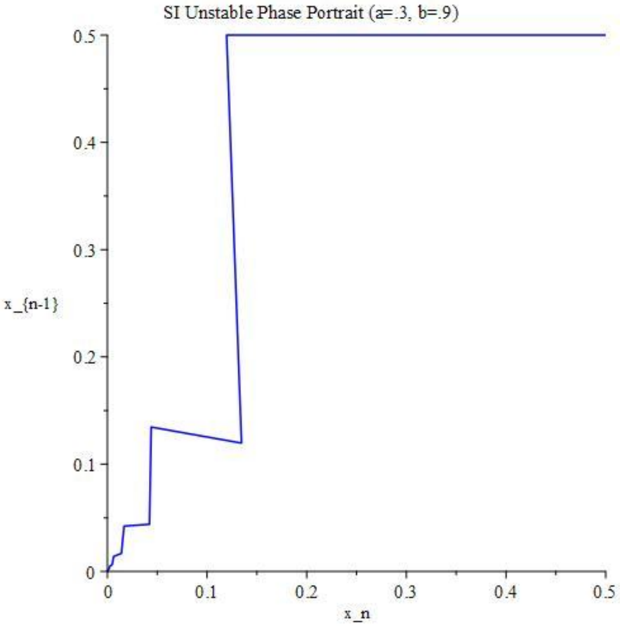


R = a/b = 0.8889

- Virus dies out of population

Conjecture 2 Graphs (Unstable)

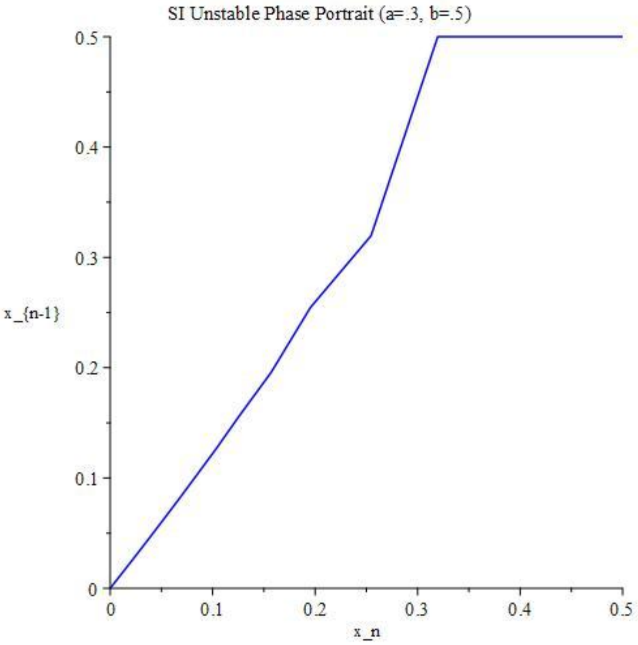


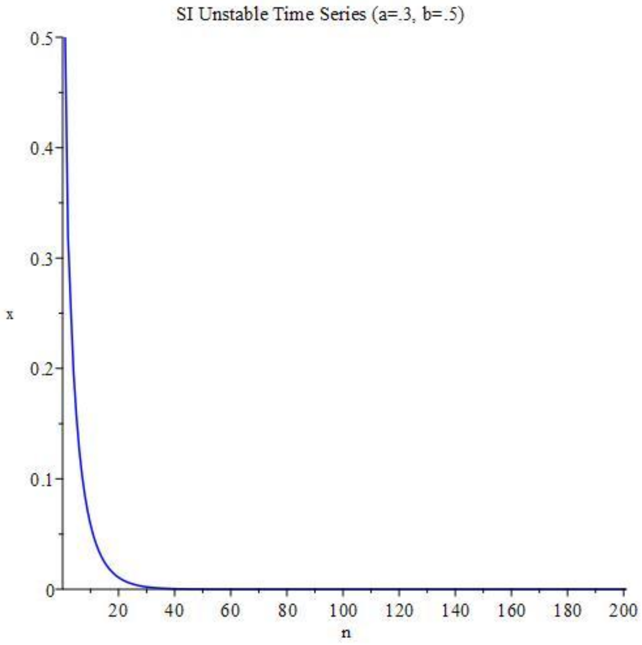


R = a/b = 0.3333

- Virus dies out of population

Conjecture 2 Graphs (Unstable)





R = a/b = 0.6

- Virus dies out of population