

Lecture Notes for Lecture 9 of Dr. Z.'s Dynamical Systems in Biology

The **gambler's ruin problem**, a famous problem in probability theory: is it also important in biology. See

<https://sites.math.rutgers.edu/~zeilberg/Bio25/bulte.pdf>

This is an advanced article, that you are welcome to read (or at least look at), but we will not cover it in class. But it is based on the gambler's ruin problem, so it justifies including this topic in this class. There are also other applications to biology.

In order to study the gambler's ruin problem, we need more theory. We already know how to solve **initial values linear recurrence equations** with constant coefficients where you are given a linear recurrence (in standard form)

$$a(n+k) = c_1 a(n+k-1) + c_2 a(n+k-2) + \dots + c_k a(n)$$

and **initial conditions** $a(0) = a_0, a(1) = a_1, \dots, a(k-1) = a_k$ and you have to find an explicit formula for $a(n)$ for *any* future n .

But sometimes the biological quantity is confined to a finite time $0 \leq n \leq L$, and you given some values at the begining and some values at the end. For the sake of simplicity from now we will only treat **second order boundary value problem** .

How to Solve second-order linear recurrences with constant coefficients Boundary Value Problems?

Input: A second-order (homog) linear recurrence,

$$a(n+2) = c_1 a(n+1) + c_2 a(n) \quad ,$$

assumed to hold for $0 \leq n \leq L-2$, for some positive integer L (that is also part of the input) and **initial** and **final** values

$$a(0) = a_0 \quad , \quad a(L) = a_L \quad ,$$

for some numbers a_0 and a_L .

Output: An explicit expression for $a(n)$ for all $0 < n < L$.

The steps are similar to the initial value case, but with a twist.

Step 1: Set up the **characteristic** equation

$$z^2 - c_1 z - c_2 = 0 \quad ,$$

and solve it. If you are lucky you get two **distinct** roots, let's call then z_1, z_2 . In that case the **template** for the **general solution** is

$$a(n) = A_1 z_1^n + A_2 z_2^n \quad .$$

If you have a **double root**, z_1, z_1 , the template is

$$a(n) = (A_1 n + A_2) z_1^n \quad ,$$

where A_1 and A_2 are numbers to **be determined**.

Step 2: Take advantage of the initial and final conditions, and set-up a system of two equations for the two unknowns

$$A_1 + A_2 = a_0 \quad , \quad A_1 \cdot z_1^L + A_2 \cdot z_2^L = a_L \quad ,$$

or in the case of a double-root,

$$A_2 = a_0 \quad , \quad (A_1 L + A_2) z_1^L = a_L \quad .$$

Step 3: Solve this system, getting expression for A_1 and A_2 that depend on L .

Step 4: Go back to the template and plug-in these specific values for A_1 and A_2 .

Problem 9.1: Solve the boundary-value linear recurrence (for an **arbitrary** L)

$$a(n+2) = a(n+1) - \frac{2}{9} a(n) \quad , 0 \leq n \leq L-2 \quad ,$$

subject to the boundary conditions

$$a(0) = 0 \quad , \quad a(L) = 1 \quad .$$

Sol. of problem 9.1:

The **characteristic equation** is:

$$z^2 - z + \frac{2}{9} = 0 \quad .$$

Factorizing we get

$$(z - \frac{1}{3})(z - \frac{2}{3}) = 0 \quad .$$

So there are two distinct roots $z_1 = \frac{1}{3}$ and $z_2 = \frac{2}{3}$. Hence the **general solution** is

$$a(n) = A_1 \left(\frac{2}{3}\right)^n + A_2 \left(\frac{1}{3}\right)^n \quad .$$

Plugging in $n = 0$ and $n = L$ we have

$$A_1 + A_2 = 0 \quad , \quad A_1 \left(\frac{2}{3}\right)^L + A_2 \left(\frac{1}{3}\right)^L = 1 \quad .$$

From the first equation we get $A_1 = -A_2$ hence

$$A_1 \left(\left(\frac{2}{3}\right)^L - \left(\frac{1}{3}\right)^L \right) = 1 \quad .$$

Hence

$$A_1 = \frac{1}{\left(\frac{2}{3}\right)^L - \left(\frac{1}{3}\right)^L}$$

$$A_2 = -\frac{1}{\left(\frac{2}{3}\right)^L - \left(\frac{1}{3}\right)^L}$$

Going back to the general solution we have

Ans. of problem 9.1:

$$a(n) = \frac{\left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)^n}{\left(\frac{2}{3}\right)^L - \left(\frac{1}{3}\right)^L} \quad .$$

How to solve Inhomogeneous Linear Recurrence Boundary Value Problems?

Now you also have to find a **particular solution**, PS , and then the general solution is

$$a(n) = A_1 z_1^n + A_2 z_2^n + PS \quad ,$$

otherwise it is the same, but the details are even more tedious.

Applications to Gambler's Ruin

Fair Coin:

You enter a **fair** casino (that does not exist in real life) where at each time step you win a dollar with probability $\frac{1}{2}$ or lose a dollar with probability $\frac{1}{2}$. You start with n dollars and you must exit the casino as soon as you have L dollars in your pocket, or are broke, with 0 dollars in your pocket.

Question 1: What is your probability of exiting a winner?

Question 2: What is the expected number of time-steps until you exit the casino, either as a winner or loser?

Answer to Question 1: $\frac{n}{L}$.

Proof: Let $a(n)$ be the probability that you will exit a winner if you currently have n dollars.

When you toss the coin

- With prob. $\frac{1}{2}$ you now have $n + 1$ dollars, and by definition, conditioned on you having just won a dollar, your probability of exiting a winner is $a(n + 1)$

- With prob. $\frac{1}{2}$ you now have $n - 1$ dollars, and by definition, conditioned on you having just lost a dollar, your probability of exiting a winner is $a(n - 1)$

Combining these two (mutually exclusive) events, we get a **linear recurrence**

$$a(n) = \frac{1}{2}a(n+1) + \frac{1}{2}a(n-1) \quad ,$$

with **boundary conditions**

$$a(0) = 0 \quad , \quad a(L) = 1 \quad .$$

After you put it in standard form, the characteristic equation is

$$z^2 - 2z + 1 = 0.$$

Factoring, we get $(z - 1)^2 = 0$, so we have a **double root** 1, 1, so the template is

$$a(n) = A_1 n + A_2 \quad .$$

Plugging-in $n = 0$ and $n = L$ and using $a(0) = 0$ and $a(L) = 1$ we get the equations

$$\{0 = A_2 \quad , \quad 1 = A_1 \cdot L \quad \},$$

whose solution is $A_2 = 0$, $A_1 = \frac{1}{L}$. Hence we found

Important Formula: The probability of exiting a winner if you currently have n dollars is $\frac{n}{L}$.

Answer to Question 2: $n(L - n)$ quad .

Proof: Let $b(n)$ be the expected number of rounds until you exit.

When you toss the coin

- With prob. $\frac{1}{2}$ you now have $n + 1$ dollars, and by definition, conditioned on you having just won a dollar, your expected remaining number of rounds is $\frac{1}{2}b(n + 1)$.
- With prob. $\frac{1}{2}$ you now have $n - 1$ dollars, and by definition, conditioned on you having just lost a dollar, your expected remaining number of rounds is $\frac{1}{2}b(n - 1)$.

Combining these two (mutually exclusive) events, and *adding* 1, we get the **inhomogenous linear recurrence**

$$b(n) = \frac{1}{2}b(n+1) + \frac{1}{2}b(n-1) + 1 \quad ,$$

with **boundary conditions**

$$b(L) = 0 \quad , \quad b(0) = 0 \quad .$$

(since if you have 0 dollars you have 0 rounds left, and also if you L dollars, you have 0 rounds left.

We can use the standard technique, as above, but it is faster just to verify that

$$b(n) = n(L - n) \quad ,$$

satisfies the in-homog. recurrence and the two boundary conditions. Do it!

Loaded Coin

Suppose that now the coin is loaded.

- with probability p you win dollar.
- with probability $q = 1 - p$ you lose a dollar.

Question : What is your probability of exiting a winner?

Answer

$$\frac{1 - (q/p)^n}{1 - (q/p)^L}$$

Proof: Now the recurrence is

$$a(n) = pa(n+1) + (1-p)a(n-1) \quad .$$

The characteristic equation is

$$pz^2 - z + (1-p) = 0 \quad .$$

This factorizes

$$(z-1)(pz+p-1) = 0 \quad ,$$

and we get two roots $z_1 = 1$ and $z_2 = \frac{1-p}{p} = \frac{q}{p}$

Hence the general solution is

$$a(n) = A_1 + A_2 \left(\frac{q}{p}\right)^n \quad .$$

Plugging $n = 0$ and $n = L$ we get

$$0 = A_1 + A_2 \quad , \quad A_1 + A_2 \left(\frac{q}{p}\right)^L = 1 \quad .$$

From the first equation $A_2 = -A_1$ hence

$$A_1(1 - (\frac{p}{q})^L) = 1 \quad .$$

Hence

$$A_1 = \frac{1}{1 - (\frac{p}{q})^L} \quad ,$$

$$A_2 = -\frac{1}{1 - (\frac{p}{q})^L} \quad ,$$

Combining we have

$$a(n) = \frac{1 - (q/p)^n}{1 - (q/p)^L} \quad .$$

Problem 9.2 : You enter a gambler's ruin casino where the maximum amount is 1000. You currently have 500 dollars.

- (i) What is the probability that you will exit a winner in a fair casino?
- (ii) What is your probability of exiting a winner if your prob. of winning is slightly less, 0.49 (you can use a calculator or Maple)
- (iii) What is your probability of exiting a winner if your prob. of winning is very slightly less, 0.499 (you can use a calculator or Maple)

Ans. to 9.2: (i) $\frac{500}{1000} = 0.5$

(ii)

$$\frac{1 - (0.51/0.49)^{500}}{1 - (0.51/0.49)^{1000}} = 0.000000002055662751$$

(iii)

$$\frac{1 - (0.501/0.499)^{500}}{1 - (0.501/0.499)^{1000}} = 0.1192026437$$

Moral: Never enter a casino! Even if your chance of winning is ever-so-slightly less than $\frac{1}{2}$ your prob. of exiting a winner is only %11.92 as opposed to %50.