Lecture Notes for Lecture 7 of Dr. Z.'s Dynamical Systems in Biology

These notes are based on sections 2.1 and 2.2 of Leah Edelstein-Keshet's excellent textbook:

`https://sites.math.rutgers.edu/~zeilberg/Bio25/keshet/keshet2.pdf` .

You should read these two sections carefully.

Quick overview of Fundamental Notions

Given *any* function defined on the real numbers, $f(x)$, the corresponding **recurrence** is

$$x(n+1) \,=\, f(x(n)) \quad .$$

You are also given an *initial condition* $x(0)$.

Unless $f(x) = ax + b$ for some numbers a,b, this is a **non-linear recurrence**.

Important concept: The **orbit** of a non-linear recurrence starting at $x(0)$, of length k is the sequence of numbers

$$[x(0), x(1), x(2), \ldots, x(k-1)]$$

In other words

$$[x(0), f(x(0)), f(f(x(0))), f(f(f(x(0)))), \ldots].$$

Problem 7.1: Find the orbit of length 5 starting at $x(0) = \frac{1}{2}$ of the recurrence

$$x(n+1) \,=\, 2\,x(n)$$

Solution of 7.1:

$$[\frac{1}{2}, 1, 2, 4, 8]$$

Problem 7.2: Find the orbit of length 5 starting at $x(0) = \frac{1}{4}$ of the recurrence

$$x(n+1) \,=\, 2\,x(n)\,(1 - x(n)) \quad .$$

Solution of 7.2:

$$x(0) = \frac{1}{4} \quad ,$$

$$x(1) = 2 \cdot \frac{1}{4} \cdot (1 - \frac{1}{4}) = \frac{3}{8} \quad ,$$

$$x(2) = 2 \cdot \frac{3}{8} \cdot (1 - \frac{3}{8}) = \frac{15}{32} \quad ,$$

$$x(3) = 2 \cdot \frac{15}{32} \cdot (1 - \frac{15}{32}) = \frac{255}{512} \quad ,$$

$$x(4) = 2 \cdot \frac{255}{512} \cdot (1 - \frac{255}{512}) = \frac{65535}{131072} \quad ,$$

Ans. to 7.2:

$$[\frac{1}{2}, \frac{3}{8}, \frac{15}{32}, \frac{255}{512}, \frac{255}{512}, \frac{65535}{131072}] \quad .$$

Important Concept A number c that is a fixed-point of $x \rightarrow f(x)$ is called a **steady-state**.

i.e. c satisfies

$$f(c) = c \quad .$$

Note that if $x(0) = c$ then $x(1) = c$, $x(2) = c$, etc. and the orbit of any length consists of only c.

The steady-state of a linear first-order recurrence

If $x(n+1) \ = \ ax(n) + b$

then to get the steady-state we solve

$$c \ = \ a\,c + b$$

solving for c we get

Fact: If $a \neq 1$, then the only steady-state of $x(n+1) \ = \ ax(n) + b$ is $c \ = \ \frac{b}{1-a}$

Note that for the stady-state $c \ = \ \frac{b}{1-a}$, we have:

$$(x(n+1) \ - \ c) \ = \ a \cdot (x(n) \ - \ c) \quad .$$

Hence

Important Fact: For a linear recurrence $x(n+1) \ = \ ax(n) + b$

• if $|a| < 1$, then the distance of the members of the orbit, starting anywhere, keep **shrinking** and eventually the orbit converges to the steady-state $c = \frac{b}{1-a}$. This is the case of a **stable** steady-state.

• if $|a| > 1$, then the distance of the members of the orbit, starting anywhere, keeps growing exponentially, and the sequence diverges.

This is the case of an **unstable** steady-state.

How to Find all the steady-states of a non-linear recurrence, and to determine their stability

If it is linear, we already know how to do it. Otherwise consider the non-linear recurrence

$$x(n+1) \;=\; f(x(n)) \quad .$$

Step 1: Solve the (algebraic, or trig, or whatever) equation

$$x = f(x) \quad .$$

Discard all the complex roots (we live in a real world) and let them be

$$z_1, z_2, \ldots, z_k$$

These are the steady-states.

To determine stability, first compute $f'(x)$ and for each z_i

• If $|f'(z_i)| < 1$ then z_i is a **stable** steady-state.

• If $|f'(z_i)| > 1$ then z_i is an **unstable** steady-state.

• If $|f'(z_i)| = 1$ then z_i is a **semi-stable** steady-state.

Explanation : By Taylor's expansion $f(z_i+h) - f(z_i) = f'(z_i)\,h + (f''(z_i)/2)\cdot h^2 + \ldots$. For small h only the first term is significant. If $|f'(z_i)| < 1$ then it is like a linear recurrence with $|a| < 1$ so the distances to the steady-state shrinks as you keep iterating the function. If $|f'(z_i)| > 1$ then it is like a linear recurrence $x(n+1) \;=\; a\,x(n) \;+\; b$ with $|a| > 1$ so the distances to the steady-state explodes as you keep going. If $|f'(z_i)| = 1$ then it is none of the above.

Problem 7.3

For the non-linear recurrence

$$x(n+1) = \frac{1}{4}x(n)^3 - 3x(n)^2 + \frac{51}{4}x(n) - 15$$

(i) Verify that the following three points are **steady-states**: $x = 3$, $x = 4$, $x = 5$.

(ii) For each of them decide whether they are stable or not.

(iii) For one unstable, and one stable steady-state, take a number close to it, and compute the first six members of the orbits (you can use Maple or a calculator). See if it seems to converge to the steady-state or runs away from it.

Sol. to 7.3:

The underlying function is

$$f(x) = \frac{1}{4}x^3 - 3x^2 + \frac{51}{4}x - 15 \quad .$$

We have $f(3) = 3$, $f(4) = 4$, $f(5) = 5$ (check!)

Now

$$f'(x) = \frac{3}{4}x^2 - 6x + \frac{51}{4} \quad .$$

When $x = 3$ we have

$$f'(3) = \frac{3}{2} \quad .$$

Since $|\frac{3}{2}| > 1$, we have: $x = 3$ is an **unstable** steady-state.

When $x = 4$ we have

$$f'(4) = \frac{3}{4} \quad .$$

Since $|\frac{3}{4}| < 1$, we have: $x = 4$ is an **unstable** steady-state.

When $x = 5$ we have

$$f'(5) = \frac{3}{2}$$

Since $|\frac{3}{2}| > 1$, we have: $x = 5$ is an **unstable** steady-state.

(iii) In Maple define

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f:=x -> 1/4*x**3 -3*x**2+51/4*x-15;
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For the unstable steady-state $x = 3$, let's take $x(0) = 3.1$. We have

$$x(1) = f(3.1) = 3.14275000$$

$$x(2) = f(3.14275000) = 3.19956905 \quad ,$$

$$x(3) = f(3.19956905) = 3.27146981 \quad ,$$

$$x(4) = f(3.27146981) = 3.35693437 \quad ,$$

$$x(5) = f(3.35693437) = 3.45121851 \quad .$$

You see that indeed as you keep going the terms of the recurrence go further and further from $x = 3$.

For the stable steady-state $x = 4$, let's take $x(0) = 4.1$. We have

$$x(1) = f(4.1) = 4.07525000$$

$$x(2) = f(4.075250004) = 4.05654404 \quad ,$$

$$x(3) = f(4.05654404) = 4.04245322 \quad ,$$

$$x(4) = f(4.04245322) = 4.03185904 \quad ,$$

$$x(5) = f(4.03185904) = 4.02390236 \quad .$$

You can see that the terms of the sequence get closer-and-closer to the steady-state $x = 4$, that confirms that it is steady.