Lecture Notes for Lecture 5 of Dr. Z.'s Dynamical Systems in Biology

Standard Form of a Linear Recurrence equation

The **standard format** of a homogeneous linear recurrence with constant coefficients is, of order k

$$a(n+k) = c_1 a(n+k-1) + c_2 a(n+k-2) + \ldots + c_k a(n) \quad ,$$

for some constants $c_1, \ldots, c_k$.

Any recurrence not in standard form can be brought to that form.

Problem 5.1:

Consider the recurrence, with initial conditions

$$a(n+2) - 3a(n+1) + a(n) + a(n-2) = 0 \quad , \quad a(0) = 1, a(1) = 2, a(2) = 3, a(3) = -1 \quad .$$

Convert if to standard form.

Sol. to Problem 5.1 look at the largest i such $a(n-i)$ shows up in the left side and replace n by $n+i$. In this problem, because that $a(n-2)$ shows up (but **not** $a(n-3)$, $a(n-4)$ etc.), $i = 2$. So we replace n by $n+2$ everywhere.:

$$a(n+4) - 3a(n+3) + a(n+2) + a(n) = 0 \quad , \quad a(0) = 1, a(1) = 2, a(2) = 3, a(3) = -1 \quad .$$

Now move $a(n+4)$ to the left and everything else to the right, getting:

Ans. to 5.1:

$$a(n+4) = 3a(n+3) - a(n+2) - a(n) \quad , \quad a(0) = 1, a(1) = 2, a(2) = 3, a(3) = -1 \quad .$$

First-Order Vector Recurrence

The format of a first-order vector recurrence (with constant coefficients) is, where $\mathbf{x}$(n) is a sequence of **vectors** (of dimension k say)

$$\mathbf{x}(n+1) = A\,\mathbf{x}(n) \quad , \quad \mathbf{x}(0) = \mathbf{x}_0 \quad ,$$

where A is a $k \times k$ matrix of numbers, and $\mathbf{x}_0$ is some column vector with k components.

Problem 5.2: Consider the vector recurrence

$$\mathbf{x}(n+1) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{x(n)} \quad , \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad ,$$

use Maple to find $\mathbf{x}(10)$.

Sol. of 5.2 It is easy to see that $\mathbf{x}(n) = A^n \mathbf{x}(0) = A^n \mathbf{x}_0$, where $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Our matrix A is

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

typing (after `with(linalg):`)

`evalm(A**10)` gives you

$$A^{10} = \begin{bmatrix} 29525 & 29524 \\ 29524 & 29525 \end{bmatrix} \quad .$$

We now multiply it by $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ getting

$$\begin{bmatrix} 29525 & 29524 \\ 29524 & 29525 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 147622 \\ 147623 \end{bmatrix} \quad .$$

Ans, to 5.2:

$$\mathbf{x}(10) = \begin{bmatrix} 147622 \\ 147623 \end{bmatrix} \quad .$$

How to convert a k-th order linear recurrence to first-order vector recurrence?

Since Maple (and Matlab) are so good with exponentiating matrices, it is more efficient to convert a linear recurrence of order k to a vector recurrence of the format

$$\mathbf{a}(n+1) = A\,\mathbf{a}(n)$$

that has the same information. It is a neat trick.

Consider the linear recurrence in **standard form** (if it is not you must first bring it to that form like above).

$$a(n+k) = c_1 a(n+k-1) + c_2 a(n+k-2) + \ldots + c_k a(n) \quad .$$

Define

$$\mathbf{a}(n) = \begin{bmatrix} a(n+k-1) \\ a(n-1) \\ \ldots \\ a(n) \end{bmatrix}$$

Now define a $k \times k$ matrix, let's call it A, whose first row is

$$[c_1 \quad c_2 \quad c_3 \quad \ldots \quad c_{k-1} \quad c_k]$$

Its second row is

$$[1 \quad 0 \quad 0 \quad \ldots \quad 0 \quad 0]$$

Its third row is

$$[0 \quad 1 \quad 0 \quad \ldots \quad 0 \quad 0]$$

$$\ldots$$

$$\ldots$$

The last row (row k) is:

$$[0 \quad 0 \quad 0 \quad \ldots \quad 1 \quad 0]$$

Convince yourself that the original information is equivalent to the **first-order** vector recurrence

$$\mathbf{a}(n+1) \;=\; A\,\mathbf{a}(n) \quad ,$$

with initial condition

$$\mathbf{a}(0) = \begin{bmatrix} a(k-1) \\ \ldots \\ a(0) \end{bmatrix} \quad .$$

Problem 5.3: Convert the recurrence

$$a(n+2) = a(n+1) + 2a(n) \quad , \quad a(0) = 1 \quad , \quad a(1) = 5 \quad ,$$

into a vector first-order recurrence and use Maple to find $a(100)$.

Sol. to 5.3 The vector version is

$$\mathbf{a}(n+1) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \mathbf{a}(n) \quad .$$

We have the initial condition (in vector form)

$$\mathbf{a}(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad .$$

The matrix is

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \quad .$$

Hence

$$\begin{bmatrix} a(101) \\ a(100) \end{bmatrix} = \mathbf{a}(100) = A^{100}\mathbf{a}(0) = A^{100} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

In Maple you type

```
evalm(A**(100)&*matrix([[5],[1]]));
```

getting

$$\mathbf{a}(100) = \begin{bmatrix} a(101) \\ a(100) \end{bmatrix} = \begin{bmatrix} 507060240091291760598681282150 5 \\ 253530120045645880299340641075 1 \end{bmatrix}$$

Hence

Ans. to 5.3:

$$a(100) = 2535301200456458802993406410751 \quad .$$

Leslie Matrix and Population Growth

There are $A+1$ age-groups with initially, at time $t=0$

- $n_0(0)$ 0-year-olds

- $n_1(0)$ 1-year-olds

...

- $n_A(0)$ A-year-olds

Beyond age $A+1$, the females are no longer fertile.

Let $n_i(t)$ be the expected number of i-year-olds after t years.

The **fertility** rate is the vector $[f_0, f_1, \ldots, f_A]$ that tells you that every i-year-old female gives birth to f_i new-borns (usually $f_0 = 0$, but in some species 0-year-old can make new babies).

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$$n_0(t+1) = f_0 n_0(t) + f_1 n_0(t) + \ldots + f_A n_A(t) \quad ,$$

since, for each $i = 0, 1, \ldots, A$, each and every one of the $n_i(t)$ i-year-olds makes f_i new babies and hence altogether $f_i n_i(t)$. Now add them up.

There is also a **survival** vector $p_0, \ldots, p_{A-1}$ such that p_i is the probability of an i-year-old making it to next year.

Recall that $n_0(t), n_1(t), \ldots, n_A(t)$ is the exepected number of 0-year-old, 1-year-old, ..., A-year-olds after t years. So we have

$$n_1(t+1) = p_0 n_0(t) \quad ,$$

since only a p_0 fraction of 0-year-olds survive from year t to year $t+1$.

Similarly

$$n_2(t+1) = p_1 n_1(t) \quad ,$$

since only a p_1 fraction of 1-year-olds survive from year t to year $t+1$.

and in general

$$n_i(t+1) = p_{i-1} n_{i-1}(t) \quad ,$$

since only a p_{i-1} fraction of $(i-1)$-year-olds survive from year t to year $t+1$,

for all $0 \leq i \leq A$.

All this can be summarized with a vector **first-order** recurrence

$$\begin{bmatrix} n_0(t+1) \\ n_1(t+1) \\ \dots \\ \dots \\ n_A(t+1) \end{bmatrix} = L \begin{bmatrix} n_0(t) \\ n_1(t) \\ \dots \\ \dots \\ n_A(t) \end{bmatrix}$$

where L is the **Leslie matrix** of dimensions $(A+1) \times (A+1)$ whose

- first row is $[f_0, f_1, \dots, f_A]$
- second row is $[p_0, 0, 0, \dots, 0]$
- third row is $[0, p_1, 0, \dots, 0]$

...

- last row (row $A+1$) is $[0, 0, 0, \dots, p_{A-1}, 0]$

Problem 5.4

In a certain species only one-year-olds, two-year-olds, three-year-olds, and four-year-olds are fertile. We have

At time $t = 0$ there were

- 10 0-year-olds
- 20 1-year-olds
- 30 2-year-olds

- 40 3-year-olds
- 50 4-year-olds
- zero-year-olds can't have babies
- Every 1-year-old female makes 3 babies on average
- Every 2-year-old female makes 2 babies on average
- Every 3-year-old female makes 1 babies on average
- Every 4-year-old female makes 1 babies on average

We also know

- The probability that a zero-year-old will survive the year is 0.6
- The probability that a one-year-old will survive the year is 0.8
- The probability that a two-year-old will survive the year is 0.4
- The probability that a three-year-old will survive the year is 0.3

a. Set up the **Leslie matrix**

b. If right now there are 10 zero-year-olds, 20 one-year-olds, 30 two-year-olds, 40 three-year-olds, and 50 four-year-old, what is the expected number of 2-year-olds after two years?

Sol. to 5.4 part a

$$L = \begin{bmatrix} 0 & 3 & 2 & 1 & 1 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 \end{bmatrix}$$

Sol. to 5.4 part b

$$\begin{bmatrix} n_0(2) \\ n_1(2) \\ n_2(2) \\ n_3(2) \\ n_4(2) \end{bmatrix} = L^2 \begin{bmatrix} n_0(0) \\ n_1(0) \\ n_2(0) \\ n_3(0) \\ n_4(0) \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & 2 & 1 & 1 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 \end{bmatrix}^2 \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \\ 50 \end{bmatrix}$$

Now we go into Maple. After doing

```
with(linalg):
```

type:

```
L:=matrix([[0,3,2,1,1],[0.6,0,0,0,0],[0,0.8,0,0,0],[0,0,0.4,0,0],[0,0,0,0.3,0]]);
```

```
v:=matrix([[10],[20],[30],[40],[50]]);
```

the desired number is

```
evalm(L**2 &*v)[3,1];
```

and Maple tells you that it is: 4.80.

Ans. to 5.4 part b: The expected number of 2-year-olds after 2 years is 4.8