Lecture Notes for Lecture 3 of Dr. Z.'s Dynamical Systems in Biology

So far we know how to solve **homogeneous** linear recurrence equations and differential equations. Today we will handle the more difficult case of **inhomogeneous** equations. Recall that for a linear equation *homogeneous* means that once you put everything that depends on the unknown function and its derivatives or shifts (in the case of difference equations) the **right side** is 0. If it is not 0, then it is some function of x (or n).

For example, the recurrence

$$a(n) = a(n-1) + 3a(n-2) \quad ,$$

is homogeneous, since the very first step is to use algebra to bring it to the form

$$a(n) - a(n-1) - 3a(n-2) = 0 \quad .$$

Now the right side is 0 so it is homogeneous and you already know how to solve it.

On the other hand the recurrence

$$a(n) = a(n-1) + 3a(n-2) + n^2 \quad ,$$

is **not** homogeneous since it is equivalent to the equation

$$a(n) - a(n-1) - 3a(n-2) = n^2 \quad .$$

and **now** the right side is **not** 0.

Problem 3.1: Solve the following difference equation subject to the indicated initial conditions.

$$a(n) - 7a(n-1) + 12a(n-2) \ = \ 6n - 11 \quad , \quad a(0) = 3 \quad , \quad a(1) = 9 \quad .$$

Solutions of 3.1: First we find the **general solution** of the **homogeneous version**

$$a(n) - 7a(n-1) + 12a(n-2) = 0 \quad ,$$

(forget about the initial conditions for now), and get that the general solution is

$$a(n) = C_1 3^n + C_2 4^n \quad ,$$

(Explain why!) Next , by **undetermined coefficients** we find a **particular solution**. Since the **right hand side** is a **polynomial** of degree 1, we try out a **trial solution** (or **template**)

$$\alpha n + \beta \quad ,$$

where α, β are some numbers **to be determined**.

Plugging into the recurrence we get

$$(\alpha \cdot n + \beta) - 7(\alpha \cdot (n-1) + \beta) + 12(\alpha \cdot (n-2) + \beta) = 6n - 11$$

Simplifying

$$(6\alpha)\, n + (6\beta - 17\alpha) = 6n - 11 \quad .$$

Comparing coefficients of n and n^0 we get the system of two equations and two unknowns

$$6\,\alpha = 6 \quad , \quad 6\beta - 17\alpha = -11 \quad ,$$

giving $\alpha = 1$ and $\beta = 1$, hence a **particular solution** is

$$n + 1 \quad .$$

Hence the **general solution** of our recurrence is

$$a(n) = C_1 3^n + C_2 4^n + n + 1 \quad .$$

In order to find the constants C_1 and C_2 we need to take advantage of the **initial conditions**

$$3 = a(0) = C_1 \cdot 3^0 + C_2 \cdot 4^0 + 0 + 1 = C_1 + C_2 + 1 \quad ,$$

$$9 = a(1) = C_1 \cdot 3^1 + C_2 \dot{4}^1 + 1 + 1 = 3C_1 + 4C_2 + 2 \quad ,$$

So we have to solve the system

$$\{C_1 + C_2 + 1 = 3 \quad , \quad 3C_1 + 4C_2 + 2 = 9\}$$

that is the same as

$$\{C_1 + C_2 = 2 \quad , \quad 3C_1 + 4C_2 = 7\} \quad ,$$

whose solution is $C_1 = 1$, $C_2 = 1$. Hence the **final solution** is

Ans. to 3.1:

$$a(n) = 3^n + 4^n + n + 1 \quad .$$

Problem 3.2: Solve the following differential equation subject to the indicated initial conditions.

$$y''(x) - 7y'(x) + 12y(x) \; = \; 6x - 11 \quad , \quad y(0) = \frac{11}{8} \quad , \quad y'(0) = \frac{15}{2} \quad .$$

Solutions of 3.2: First we find the **general solution** of the **homogeneous version** (ignore the initial conditions for now)

$$y''(x) - 7y'(x) + 12y(x) \; = \; 0$$

(forget about the initial conditions for now), and get that the general solution is

$$y(x) = C_1 e^{3x} + C_2 e^{4x} \quad ,$$

(Explain why!) Next , by **undetermined coefficients** we find a **particular solution**. Since the **right hand side** is a **polynomial** of degree 1, we try out a **trial solution**

$$\alpha x + \beta \quad .$$

Plugging into the differential equation we get:

$$(\alpha x + \beta)'' - 7(\alpha x + \beta)' + 12(\alpha x + \beta) \;=\; 6x - 11$$

Simplifying:

$$-7\alpha + 12\alpha x + 12\beta = 6x - 11 \quad .$$

Simplifying more

$$(12\alpha)x + (12\beta - 7\alpha) = 6x - 11 \quad .$$

Comparing coefficients of x and x^0 we get the system of two equations and two unknowns

$$12\alpha = 6 \quad , \quad ,12\beta - 7\alpha = -11$$

giving $\alpha = \frac{1}{2}$ and $\beta = -\frac{5}{8}$, hence a **particular solution** is

$$\frac{1}{2}\,x - \frac{5}{8} \quad . \quad .$$

Hence the **general solution** of our (original, inhomogeneous) differential equation is

$$y(x) = C_1 e^{3x} + C_2 e^{4x} + \frac{1}{2}\,x - \frac{5}{8} \quad .$$

For future reference, we also need the derivative:

$$y'(x) = 3C_1 e^{3x} + 4C_2 e^{4x} + \frac{1}{2} \quad .$$

In order to find the constants C_1 and C_2 we need to take advantage of the **initial conditions**

$$\frac{11}{8} = y(0) = C_1 + C_2 - \frac{5}{8} \quad ,$$

$$\frac{15}{2} = y'(0) = 3C_1 + 4C_2 + \frac{1}{2} \quad .$$

So we have to solve the system

$$\{C_1 + C_2 = 2 \quad ,3C_1 + 4C_2 = 7 \quad .$$

whose solution is $C_1 = 1$, $C_2 = 1$. Hence the **final solution** f

Ans. to 3.2:

$$y(x) \;=\; e^{3x} \;+\; e^{4x} + \frac{1}{2}\,x - \frac{5}{8} \quad .$$