Lecture Notes for Lecture 15 of Dr. Z.'s Dynamical Systems in Biology

These notes are based on section 3.6 of Leah Edelstein-Keshet's classic "Mathematical Models in Biology", and ex. problem 18 (pp. 107-108)

The Hardy-Weinberg Law

There are two **alleles** denoted by a and A, and they are passed down from one generation to the next. A given individual can have one of three combinations: AA, aa, and Aa.

If there are N individuals, then there are $2N$ alleles. then the

Let

- Let p be the frequency of allele A
- Let q be the frequency of allele a
- u be the frequency of AA genotype
- v be the frequency of Aa genotype
- w be the frequency of aa genotype

Of course

$$p = u + \frac{1}{2}v \quad , \quad q = \frac{1}{2}v + w \quad ,$$

Under random mating we have the following frequency mating matrix where the rows are the Mothers and the columns are the fathers

$$\begin{array}{ccccc} & & AA & Aa & aa \\ -- & -- & -- & -- & -- \\ AA & | & u^2 & uv & uw \\ Aa & | & vu & v^2 & vw \\ aa & | & wu & wv & w^2 \end{array}$$

How to find the types of the offsprings?

If a *genotype* consists of two *alleles*, and there are two types of alleles a and A then the possible alleles are AA, Aa, aA, and aa, but aA and Aa are conisered the same. It is asummed that each parent is equally likley to contribute one of his or her alleles.

For example, if the father is Aa and the mother is aa

the prob. of Aa, Aa, aa, aa, are each $\frac{1}{4}$. Combining the prob. of Aa is $\frac{1}{2}$ and the prob. of aa is $\frac{1}{2}$.

Problem 15.1: Suppose that the genotypes has two alleles, but there are three types of alleles, a, b, c. Suppose the father has genotype ac and the mother has genotype ab. Assuming that each allele is equally likely, what is the prob. distribution of the offsprings?

Sol. to 15.1:

The picture is

$$\begin{matrix} a & c \\ a & b \end{matrix}$$

With prob. $\frac{1}{4}$ each the children will be of type aa, ab, ac, bc. But ca is the same as ac and cb is the same as bc so

Going back to Hardy-Weinberg, we have:

• If both parents are AA (prob. u^2) then the frequency of the offsprings being AA, Aa, aa are:

u^2, 0, 0 respectively.

• If one parent is AA and the other parent is Aa (prob. $2uv$) then the frequency of the offsprings being AA, Aa, aa are:

uv, uv, 0 respectively.

• If one parent is AA and the other parent is aa (prob. $2uw$) then the frequency of the offsprings being AA, Aa, aa are:

0, $2uw$, 0 respectively.

• If both parents are Aa (prob. v^2) then the frequency of the offsprings being AA, Aa, aa are:

$v^2/4$, $v^2/2$, $v^2/4$ respectively.

• If one parent is Aa and the other parent is aa (prob. $2vw$) then the frequency of the offsprings being AA, Aa, aa are:

0, vw, vw respectively.

• If both parents are aa (prob. w^2) then the frequency of the offsprings being AA, Aa, aa are:

0, 0, w^2 respectively.

So if the frequencies of AA, Aa, aa in this generations are u, v, w respectively, then the frequencies of these in the next generation are:

$$u^2 + uv + \frac{1}{4}v^2 \quad , \quad uv + 2uw + \frac{1}{2}v^2 + vw \quad , \quad \frac{1}{4}v^2 + vw + w^2 \quad .$$

So we have the transformation (recall that $u+v+w=1$)

$$(u,v,w) \to (u^2+uv+\frac{1}{4}v^2 \, , \, uv+2uw+\frac{1}{2}v^2+vw \, , \, \frac{1}{4}v^2+vw+w^2)$$

So if in generation n the frequencies of AA, Aa and aa were u_n, v_n, w_n respectively, then we have the vector first-order recurrence

$$u_{n+1} = u_n^2 + u_n v_n + \frac{1}{4}v_n^2 \quad ,$$

$$v_{n+1} = u_n v_n + 2u_n w_n + \frac{1}{2}v_n^2 + v_n w_n \quad ,$$

$$w_{n+1} = \frac{1}{4}v_n^2 + v_n w_n + w_n^2 \quad .$$

Maple (or you) can show that if $u_n+v_n+w_n=1$ then $u_{n+1}+v_{n+1}+w_{n+1}=1$ (it also follows from probability considerations). So this is really a 2-variable discrete dynamical system with underlying transformation

$$(u,v) \to (u^2+uv+\frac{1}{4}v^2 \, , \, uv+2u(1-u-v)+\frac{1}{2}v^2+v(1-u-v))$$

Expanding we get that the **Hardy-Weinberg Transform** is

$$(u,v) \to \left(u^2+vu+\frac{1}{4}v^2, -2vu-2u^2+2u-\frac{1}{2}v^2+v\right) \quad .$$

Question: Is there a steady-state? Is it stable? In other words in the long-run, do the relative frequencies of the three genotypes converge?

Surprise: You don't have to wait for the 'long run'. If you apply this transformation twice you get the same thing!

In other words, the second generation will have different frequencies of the three genotypes, but after **one generation** it will stay the **same** for ever after!

This is the Hardy-Weinberg law.

Generalized Hardy-Weinberg

If different genotypes have preferences, then there is still a stable steady-state but not after one iteration. See procedure `HWg(u,v,M)` in the Maple package `DMB.txt`: