

Lecture Notes for Lecture 11 of Dr. Z.'s Dynamical Systems in Biology

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The logistic map and advancement to Chaos

The (discrete) logistic equation

$$a(n+1) = r \cdot a(n) \cdot (1 - a(n)) \quad ,$$

modeling a population of a biological species with **fertility** factor r , but due to **competition** for food and other resources, has a **damping factor** that gets smaller and smaller as the population gets larger, and is assumed to be $(1 - a(n))$. (It is assumed that the maximal occupancy of the habitat is 1).

For any r there are two steady-states. The underlying function is $f(x) = rx(1 - x)$. Solving $x = f(x)$ we get

$$x = rx(1 - x) \quad .$$

Moving everything to the left we get

$$x - rx(1 - x) = 0 \quad .$$

Factoring

$$x(1 - r(1 - x)) = 0$$

Simplifying

$$x(rx - (r - 1)) = 0 \quad .$$

We get two steady-states

$$x = 0 \quad , \quad x = \frac{r - 1}{r} \quad .$$

Since everything takes place in $0 < x < 1$, only $x = 0$ makes sense when $r < 1$. Let's see whether or not it is a stable steady-state.

We have, since $f(x) = rx - rx^2$

$$f'(x) = r - 2rx \quad .$$

and we have, since $f'(0) = r$, and since right now, $0 < r < 1$, we get that $|f'(0)| < 1$, and we have the following fact.

Important Fact: The only stable steady-state of the logistic recurrence $a(n+1) = ra(n)(1 - a(n))$, when $0 < r < 1$, is $x = 0$. No matter what the initial population was, it is doomed to extinction. For any x_0 between 0 and 1 the orbit goes to 0. So for $0 < r > 1$, $x = 0$ is the only stable steady-state.

When $r > 1$, $x = 0$ is no longer a stable steady-state, since $|f'(0)| = r$ is larger than 1. Let's see for what values of r , the other steady-state, namely $x = \frac{r-1}{r}$, is stable.

We have

$$f'\left(\frac{r-1}{r}\right) = r - 2r \cdot \frac{r-1}{r} = 2 - r \quad .$$

So when

$$|2 - r| < 1 \quad ,$$

$x = \frac{r-1}{r}$ is the only stable steady-state, i.e. when .

$$1 < r < 3 \quad .$$

Important Fact: When $1 < r < 3$ the only stable steady-state is $x = \frac{r-1}{r}$ and every orbit, regardless of the starting point x_0 between 0 and 1, in the long-run, converges to that number. So the population of the species stabilizes and eventually doesn't change from one year to the next.

But when $r > 3$, there are no stable steady states, so what happens? If you run for example, in `DMB8.txt` the orbit when $r = 3.2$, starting at $x_0 = 0.3$

```
Orb(3.2*x*(1-x),x,0.3,10000,10010);
```

you get

```
[0.7994554906, 0.5130445091, 0.7994554906, 0.5130445091, 0.7994554906, 0.5130445091,
0.7994554906, 0.5130445091, 0.7994554906, 0.5130445091, 0.7994554906]
```

If you start at $x_0 = 0.7$, typing

```
Orb(3.2*x*(1-x),x,0.7,10000,10010);
```

you get the same! So in the long run the population alternates between two values, and repeats itself, every two years. In other words the result of applying the mapping $x \rightarrow rx(1-x)$ twice, let's call it $f_1(x)$:

$$f_1(x) = f(f(x)) = -r^3x^4 + 2r^3x^3 - r^3x^2 - r^2x^2 + r^2x \quad ,$$

seems to have stable steady-states. If you enter, in `DMB8.txt`

`SS(f1,x)` you get four steady-states, let's call them $s_1(r), s_2(r), s_3(r), s_4(r)$ (of course, they depend on r)

$$[s_1(r), s_2(r), s_3(r), s_4(r)] := \left[0, \frac{r-1}{r}, \frac{\frac{r}{2} + \frac{1}{2} + \frac{\sqrt{r^2-2r-3}}{2}}{r}, \frac{\frac{r}{2} + \frac{1}{2} - \frac{\sqrt{r^2-2r-3}}{2}}{r} \right] \quad ,$$

Of course, as we already know, the first two, $s_1(r) = 0$ and $s_2(r) = \frac{r-1}{r}$ are not stable, but it is possible to show (with Maple, or other software) that for

$3 < r < 3.4494897$, $|f'_1(s_3(r))| < 1$, $|f'_1(s_4(r))| < 1$ (we will demonstrate this in the next Maple class).

Moving right along, in the range

$$3.4494897 < r < 3.5440903 \quad .$$

No matter where you start, for example $x_0 = 0.3$, $x_0 = 0.5$, $x_0 = 0.7$,

```
Orb(r*x*(1-x),x,0.3,10000,10010); , Orb(r*x*(1-x),x,0.5,10000,10010); , Orb(r*x*(1-x),x,0.7,10000,10010);
```

will converge to the **same** ultimate period-4 orbit. Upping r a little more, would give an ultimate period-8 orbit, then upping r a tiny bit more, an ultimate period-16 etc. This is called **period-doubling**.

The transition values of r , from one power-of-two ultimate period to the next one, are called **bifurcation values**. Let's call them

$$[a_1, a_2, a_3, a_4, \dots] \quad .$$

They have the meaning that for $a_i < r < a_{i+1}$ every orbit, in the long run, converges to an ultimate orbit of period 2^i .

Mitchell Feigenbaum found that the first 8 values of a_i are

$$[3, 3.4494897, 3.5440903, 3.5644073, 3.5687594, 3.5696916, 3.5698913, 3.5699340]$$

and then he discovered, numerically (and later gave a “physics proof”) the following amazing fact.

Amazing fact: there is a constant, denoted by δ , called the **Feigenbaum constant**, such that

$$\lim_{i \rightarrow \infty} \frac{a_{i+1} - a_i}{a_{i+2} - a_{i+1}} = \delta \quad ,$$

and Feigenbaum computed that:

$$\delta = 4.6692 \dots \quad .$$

When you keep raising r , the ultimate periods get to be of period 256, then 512, and before you know it, for $r > 3.571 \dots$ you get **complete chaos**.

Non-Linear Higher Order Non-Linear Recurrences

The format is

$$a(n+k) = f(a(n+k-1), \dots, a(n)) \quad ,$$

for some function of k variables, $f(x_1, \dots, x_k)$.

It also needs **initial conditions**: $a(1) = a_1, \dots, a(k) = a_k$.

Once you have the first k initial values, you can get $a(k+1)$, then $a(k+2)$, etc., etc.. Note that unlike a first-order recurrence, where today's value *only* depends on what happened yesterday, now it depends also on the day-before-yesterday (if the order (k) is ≥ 2), the value three days ago (if $k \geq 3$), etc. Sometimes this is called memory- k recurrence.

How to find the potential stable-steady-state numerically

Assuming that the orbit eventually converges to one value, z , we set $a(n), a(n+1), \dots, a(n+k)$ to a common value, let's call it z , and simply solve the algebraic equation:

$$z = f(z, z, \dots, z) \quad .$$

Problem 11.1: Find the possible long-term behavior of the third-order recurrence

$$a(n+3) = \frac{1 + a(n+2) + a(n+1) + a(n)}{2 + a(n+2) + 3a(n+1) + 7a(n)} \quad ,$$

Sol. to 11.1: We have to solve

$$z = \frac{1 + z + z + z}{2 + z + 3z + 7z} \quad ,$$

So

$$z = \frac{1 + 3z}{2 + 11z} \quad ,$$

So

$$z - \frac{1 + 3z}{2 + 11z} = 0 \quad .$$

So

$$\frac{11z^2 - z - 1}{2 + 11z} = 0 \quad .$$

Solving $11z^2 - z - 1 = 0$ we have

$$\frac{1 + 3\sqrt{5}}{2} \quad , \quad \frac{1 - 3\sqrt{5}}{2} .$$

First order Non-Linear k -dimensional Vector Recurrence

They have the format

$$\mathbf{x}(n+1) = \mathbf{F}(\mathbf{x}(n)) \quad ,$$

where $\mathbf{x}(n)$ is a (column) vector with k components, and $\mathbf{F}$ is a mapping from $R^k \rightarrow R^k$.

$$[x_1, \dots, x_k] \rightarrow [f_1(x_1, \dots, x_k), \dots, f_k(x_1, \dots, x_k)] \quad ,$$

for some **multivariable** functions, of the k variables $x_1, \dots, x_k, f_1, \dots, f_k$.

How to convert an order k recurrence to a first-order vector recurrence?

Let

$$\mathbf{x}(n) = [a(n+k-1), \dots, a(n)] \quad .$$

Then the recurrence

$$a(n+k) = f(a(n+k-1), \dots, a(n)) \quad ,$$

is equivalent to

$$\mathbf{x}(\mathbf{n} + \mathbf{1}) = \mathbf{F}(\mathbf{x}(n)) \quad ,$$

where

$$F([x_1, \dots, x_k]) = [f(x_1, \dots, x_k), x_1, x_2, \dots, x_{k-1}] \quad .$$

Problem 11.2: Convert the non-linear third-order recurrence

$$a(n+3) = \cos(a(n+2)) + e^{a(n+1)} + \sin(a(n)) \quad ,$$

to a first-order vector recurrence in R^3 .

Sol. to 11.2: Here

$$f(x_1, x_2, x_3) = \cos(x_1) + e^{x_2} + \sin(x_3) \quad .$$

Hence. setting as usual

$$\mathbf{x}(n) = [a(n+2), a(n+1), a(n)] \quad .$$

We have

$$\mathbf{x}(n+1) = \mathbf{F}(\mathbf{x}(n)) \quad .$$

where $\mathbf{F}$ is the transformation in R^3 :

$$[x_1, x_2, x_3] \rightarrow [\cos(x_1) + e^{x_2} + \sin(x_3), x_1, x_2] \quad .$$