Solutions to Basic Concepts Qualifying test

Comment; The solutions below are very detailed. If you answered correctly, but with less detail, I probably gave you full credit (unless you phrased it in such a way that makes clear that you misunderstood the concept).

1. (10 points altogether) Consider the function defined on $f : \{1, \ldots, 10\} \rightarrow \{1, \ldots, 10\}$, defined by

$$f(x) = x^3 \ (mod \quad 11)$$

(Recall that $a \quad mod \quad b$ means the remainder when you divide a by b).

(a) (5 points) Find all the fixed points. (b) (5 points) Find the trajectory that starts with $x_0 = 2$

Sol. to 1(a):

$$f(1) = 1 \quad , \quad f(2) = 8 \quad , \quad f(3) = 5 \quad , \quad f(4) = 9 \quad ,$$

$$f(5) = 4 \quad , \quad f(6) = 7 \quad , \quad f(7) = 2 f(8) = 6 \quad , \quad f(9) = 3 \quad , \quad f(10) = 10 \quad .$$

Hence the set of fixed points is $\{1, 10\}$.

Remark: A cleverer way would be to solve

$$x^3 = x \quad mod \quad 11$$

This means

$$x^3 - x$$

is divisible by 11 this means that

$$x(x-1)(x+1) \quad ,$$

is divisible by 11. Since 11 is prime this means that, modulo 11, $x = 0$ (that is not a member of our set, and hence does not count) or $x = 1$, or $x = -1 \quad mod \quad 11 = 1$

Sol. to 1(b): From the above

$$[2, 8, 6, 7, 2]$$

This is a **periodic orbit**.

2. (10 points altogether) (a) (2 points) Write down the **format** of a continuous-time dynamical system with three quantities (call them $x(t)$, $y(t)$, $z(t)$) (b) (2 points) *Define* an **equilibrium solution**. (c) (2 points) Describe how to find them. (d) (2 points) *Define* **stable equilibrium solution** (e) (2 points) Describe how to find all the stable equilibrium solutions, using calculus (not numerics)

Sol. of 2(a): For certain multivariable functions (with three variables), let's call them $f(x, y, z), g(x, y, z), h(x, y, z)$ we have

$$x'(t) = f(x(t), y(t), z(t)) \quad , \quad y'(t) = g(x(t), y(t), z(t)) \quad , \quad z'(t) = h(x(t), y(t), z(t)) \quad .$$

Sol. of 2(b): An **equilibrium solution** of the above system is a solution where $x(t), y(t), z(t)$ are each **constant functions** (of course the constants are usually different than each other). In other words there exist **numbers** a, b, c, such that the solution is: $x(t) = a$, $y(t) = b$, $z(t) = c$.

Sol. of 2(c): Since for these very simple kind of solutions the left sides are 0, replacing $x(t), y(t), z(t)$ on the right side by the numbers a, b, c, gives an **algebraic** system of equations

$$\{f(a, b, c) = 0 \quad , \quad g(a, b, c) = 0 \quad , \quad h(a, b, c) = 0\} \quad ,$$

in the unknowns $\{a, b, c\}$ that Maple can (often, but not always) solve.

Sol. of 2(d): A **stable equilibrium solution** $x(t) = a$, $y(t) = b$, $z(t) = c$ is, first of all an equilibrium solution (see above), and that has the property that for some $\delta > 0$ all (a_1, b_1, c_1) whose distance from (a, b, c) is less than δ have the property that the unique solution of the **initial value problem** of our dynamical system with initial conidtions

$$x'(0) = a_1 \quad , \quad y'(0) = b_1 \quad , \quad z'(0) = c_1 \quad ,$$

let's call this solution (with these modified initial values) $x_1(t), y_1(t), z_1(t)$, have the property that in the long run $x_1(t)$, $y_1(t)$, $z_1(t)$ get closer-and-closer to a,b, c respectively. In symbols

$$\lim_{t \to \infty} x_1(t) = a \quad , \quad \lim_{t \to \infty} y_1(t) = b \quad , \quad \lim_{t \to \infty} z_1(t) = c \quad .$$

Sol. of 2(e): First compute the **Jacobian** of the **underlying transformation**, getting a certain 3×3 matrix whose entries are functions of x, y, z. Let's call it $J(x, y, z)$. Then for each equilibrium solution found in 2(c) above, $x = a, y = b, z = c$, plug these into the general Jacobian, getting a **numerical matrix** $J(a, b, c)$. Then find all its **eigenvalues** that are, in general, complex numbers (if they happen to be real numbers, think of them as complex numbers where the imaginary part happens to be 0). If the **real parts** are **all** negative, then the candidate equilibrium solution is **stable**. Otherwise not.

3. (10 points altogether) In our own planet, the force of gravity acting on a particle of mass m, pointing towards the center of the earth is mg where g is Newton's constant. Also in our planet (and universe) the force equals the mass times the second derivative of the position.

(a) (4 points) A ball is dropped (freely, i.e. it is not pushed down or up), at time $t = 0$, from a tower of height 1000 meters. If $x(t)$ is the distance of the particle from the top of the tower at time t set up a differential equation for $x(t)$ and the initial conditions.

(b) (4 points) Solve the initial value problem, and get an explicit formula in terms of t for $x(t)$.

(c) (2 points) Find the time that it would take to reach the ground, (Leave your answer in terms of g).

Sol. of 3(a):

$$x''(t) = g \quad , \quad x(0) = 0 \quad , \quad x'(0) = 0 \quad .$$

Sol. of 3(b): $x'(t) = \int x''(t)\,dt = \int g\,dt = gt + C$. Since $x'(0) = 0$, we have $C = 0$, hence $x'(t) = gt$. $x(t) = \int x'(t)\,dt = \int gt\,dt = \frac{1}{2}gt^2 + C$. Since $x(0) = 0$, $C = 0$, hence, **ans. to 3(b)**: $x(t) = \frac{1}{2}\,gt^2$.

Sol. of 3(c): Solving $\frac{1}{2}gt^2 = 1000$, we get $t = \sqrt{\frac{2000}{g}}$.

Ans. to 3(c): It would reach the ground after $\sqrt{\frac{2000}{g}}$ seconds.

4. (10 points) A certain particle is moving in such a way that its fifth-derivative with respect to time is 1 m/s^5 At time $t = 0$ its position, and the first four derivatives of the position are 0. How far from the starting point is the particle after 2 seconds?

Sol. of 4: We have to solve the initial value differential equation

$$x^{(5)}(t) \;=\; 1 \quad , x(0) = 0, x'(0) = 0, x''(0) = 0, x'''(0) = 0, x^{(4)}(0) = 0 \quad .$$

Repeated integration (the constants of integration are always 0 gives $x(t) = \frac{t^5}{120}$. Plugging-in $t = 2$, we get

Ans. to 4: The particle would be a distance of $\frac{32}{120} = \frac{4}{15}$ meters from the starting point.

5. (10 points) In a mini-internet with two web-sites only, the probability of a random surfer, who is currently at site 1 of staying there at the next time-step is $\frac{1}{3}$ and the probability of random surfer, who is currently at site 2 of still staying there at the next time-step is $\frac{1}{5}$. In the long run, what fractions of the times are spent at each site?

Sol. of 5: Let x_1 and x_2 be the ultimate fractions of time spent at site 1 and 2 respectively. Then we have the equations

$$x_1 = \frac{1}{3}x_1 + \frac{4}{5}x_2 \quad , \quad x_2 = \frac{2}{3}x_1 + \frac{1}{5}x_2 \quad .$$

(Look at the 'traffic' flowing into site 1, it gets $\frac{1}{3}x_1$ from itself, and gets $\frac{4}{5}x_2$ from site 2. Similary for the traffic that goes into site 2.)

Simplifying (both equations give you the same thing!)

$$\frac{2}{3}x_1 = \frac{4}{5}x_2 \quad ;$$

Hence

$$x_2 = \frac{5}{6} x_1 \quad .$$

Of course $x_1 + x_2 = 1$, hence

$$x_1 + \frac{5}{6} x_1 = 1 \quad ,$$

hence

$$\frac{11}{6} x_1 = 1 \quad ,$$

hence $x_1 = \frac{6}{11}$ and $x_2 = \frac{5}{11}$.

Ans. to 5: In the long run, the random surfer would spend $\frac{6}{11}$ of the time at site 1, and $\frac{5}{11}$ at site 2.

Comment: Another solution would be the two columns of the limit as n goes to ∞ of

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}^n \quad .$$

Taking $n = 1000$ in Maple (or Matlab) would give a very good approximation. People who did it that way got 8 points out of the 10 points.

6. (10 points) For each of the following equations (or system of equations), and for each of the properties, state whether it is : (i) algebraic or differential or difference (ii) linear or non-linear

(a) $x^3 - 6x + 1 = 0$ (b) $\{x + 2y = 6, 6x + y = 11\}$ (c) $x''(t) = x'(t)^3 + x(t)$ (d) $x''(t) - 3x'(t) + 2x(t) = 0$
(e) $x(n) = x(n-1) + y(n-1) \quad , \quad y(n) = x(n-1) + 4y(n-1)$

(f) $x(n) = x(n-1)^2 + y(n-1) \quad , \quad y(n) = x(n-1) + 4y(n-1)$

Sol. to 6:

(a) algebraic; non-linear

(b) algebraic; linear

(c) differential; non-linear

(d) differential; linear

(e) difference; linear

(f) difference; non-linear

7. Consider the transformation from R^2 to R^2 given by

$$[x, y] \rightarrow [(1 - 3x - 3y)(2 - x - 2y), (2 - 3x - 2y)(1 - 3x - y)] \quad .$$

(a) (5 points) Write down the continuous-time dynamical system with the above underlying transformation.

(b) (5 points) According to `DMB.txt`, the set of equilibrium solutions is $\left\{[0,1], \left[\frac{1}{3}, 0\right], \left[\frac{4}{3}, -1\right]\right\}$.

Also according to `DMB.txt` the set of stable equilibrium solutions is $[\frac{4}{3}, -1]$.

If the initial conditions for the above dynamical system are $x(0) = \frac{4}{3}$ and $y(0) = -1$, what is the value of $x(1000)$ **exactly**?

If the initial conditions for the above dynamical system are $x(0) = 1.3$ and $y(0) = -0.9$, what is the value of $y(1000)$ **approximately**?

Sol. to 7(a):

$$x'(t) = (1 - 3x(t) - 3y(t))\,(2 - x(t) - 2y(t))\,,$$

$$y'(t) = (2 - 3x(t) - 2y(t))\,(1 - 3x(t) - y(t)) \quad .$$

Sol. to 7(b), first part: $x(1000) = \frac{4}{3}$ (once at an equilibrium solution, always there!, regardless whether it is stable or not).

Sol. to 7(b), second part: $y(1000)$ is approximately (but, most probably), **very close**, to -1 , since it is stable equilibrium, and the initial conditions given are pretty close to the equilibrium solution.

8. (10 points) 1000 chickens lay 1000 eggs in 1000 days. How many eggs do 2000 chickens lay in 2000 days?

Sol. to 8: One chicken lays $\frac{1}{1000}$ eggs in one day. Hence 2000 chickens lay $\frac{1}{1000} \cdot 2000 \cdot 2000 = 4000$ eggs.

Ans. to 8: 4000 eggs.

9. (5 points) Verify whether or not the function $x(t) = \sin t$ is a solution of the initial value problem differential equation

$$x(t)^2 + x'(t)^2 = 1 \quad , \quad x(0) = 0 \quad .$$

Sol. to 9: $x'(t) = \cos t$. Thanks to the famous trig identity $\sin^2 t + \cos^2 t = 1$ the differential equation is satisfied. Since $\sin 0 = 0$, also the initial conditions are satisfied.

10 (5 points) A woman looks at a picture on the wall and says: "Brothers and sisters have I none, but this woman's mother is my mother's daughter". At whose picture is she looking at.

Sol. to 10: Let's call the woman staring at the picture W. mother's daughter (W)= daughter(mother(W))= W (since W is an only child). Hence mother(WomanInPicture)= W

Hence

WomanInPicture= daughter of W.

Ans. to 10: The woman was looking at a portrait of her daughter.

11. (a) (5 points) Explain how you find the equilibrium solutions for discrete-time and continuous-time dynamical systems. Why are the methods different?

(b) (5 points) Explain how you find the stable equilibrium solutions for discrete-time and continuous-time dynamical systems. Why are the methods different?

Sol. of 11(a): Calling the underlying transformation $\mathbf{F} : R^k \to R^k$, For a discrete time dynamical system you solve the algebraic system

$$\mathbf{F}(\mathbf{x}) = \mathbf{x} \quad .$$

For continuous time you solve the (simpler) algebraic system

$$\mathbf{F}(\mathbf{x}) = \mathbf{0} \quad .$$

They are different, because in discrete-time $\mathbf{x} \to \mathbf{F}(\mathbf{x})$ tells you how things change from one generation to the next and at equilibrium, nothing changes.

On the other hand, in continuous time $\mathbf{F}(\mathbf{x})$ tells you the rate of change, and at equilibrium, the rate of change is $\mathbf{0}$.

Sol. to 11(b): You compute the **Jacobian**, and plug-in the examined equilibrium solution, getting a certain **numerical matrix**. Then you compute the eigenvalues.

For discrete-time: If all the eigenvalues have absolute value less than 1 it is **stable**. Otherwise not

For continuous-time: If all the eigenvalues have negative real part it is **stable**. Otherwise not.

The methods are different, due to the idea of **linearization**. For a linear discrete system $\mathbf{x} \to A\mathbf{x}$ You diagonalize A getting $A = U^{-1}DU$, where D is the diagonal matrix consisting of the eigenvalues. Then $A^n = U^{-1}D^nU$, and it goes to $\mathbf{0}$ (the (only) equilibrium for a linear system) only if all the members of D (i.e. all the eigenvalues) have absolute value less than 1.

For a continuous system, when you solve the linear system every member of the solution is a linear combination of exponentials of the form $e^{\lambda t}$, where the λ are the eigenvalues and writing $\lambda = a + ib$,

$$e^{(a+ib)t} = e^{at}(\cos bt + i \sin bt)$$

blows up if $a > 0$, but goes to 0 (very fast!) if a is negative.