Homework for Lecture 7 of Dr. Z.'s Dynamical Models in Biology class

Email the answers (either as .pdf file or .txt file) to

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by 8:00pm Monday, Sept. 27, 2021.

Subject: hw7

with an attachment hw7FirstLast.pdf and/or hw7FirstLast.txt

Also please indicate (EITHER way) whether it is OK to post

1. Use the procedure `GRt(p,i,N)` in today's Maple code

`https://sites.math.rutgers.edu/~zeilberg/Bio21/M7.txt`

to write a procedure

`EstGR(p,i,N,K)`

that runs `GRt(p,i,N) K` times, and uses it to *estimate* (by simulation) the pair

[ProbabilityOfExitingAwinner , AverageDurationOfGame] .

2. (Theoretical question)

a For a fixed N, Let $x_N(i)$ be the probability of exiting a winner in a Gambler's Ruin game with max capital N dollars and entering capital of i dollars, with a **fair coin**. Explain why the following linear (homog.) recurrence is true

$$x_N(i) = \frac{1}{2}\left(x_N(i-1) + x_N(i+1)\right) \quad ,$$

is true. Also explain why the following **boundary conditions** are true

$$x_N(0) = 0 \quad , \quad x_N(N) = 1 \quad .$$

b: Prove that the closed-form expression $y_N(i) = \frac{i}{N}$ satisfies the same recurrence and boundary conditions as $x_N(i)$, and explain why we have a beautiful explicit formula $x_N(i) = \frac{i}{N}$.

c For a fixed N, Let $E_N(i)$ be the expected number of rounds it takes to complete a Gambler's ruin game (exiting either a winner or loser) Explain why the following linear (inhomog.) recurrence is true

$$E_N(i) = \frac{1}{2}\left(E_N(i-1) + E_N(i+1)\right) + 1 \quad ,$$

is true. Also explain why the following **boundary conditions** are true

$$E_N(0) = 0 \quad , \quad E_N(N) = 0 \quad .$$

d: Prove that the closed-form expression $z_N(i) = i\,(N-i)$ satisfies the same recurrence and boundary conditions as $E_N(i)$, and explain why we have a beautiful explicit formula $E_N(i) = i(N-i)$

3. (Maple question).

Use **2** (even of you were unable to prove things, but I hope that you did) to write a procedure

`ExactFairGR(i,N)`

that outputs the pair

`[ProbabilityOfExitingAwinner , AverageDuration]`

for any i and N (of course $0 \leq i \leq N$).

See how the outputs of `ExactFairGR(i,N)` and `EstGR(1/2 , i,N,3000)` compare to each other, for $N = 20$ and all i from 1 to 19.

4. Generate 10 random stochastic matrices using `RandSM(N)` (with $N = 10$), and compare the outputs of

`StSa(P,K)`, `StSp(P,K)` (with $K = 4000$), and the exact `StS(P)`.

5. (Optional Challenge, 10 brownie points) Look up in the literature the exact formulas for the probability of exiting a winner in a Gambler's Ruin game with general probability of winning a dollar p (not necessarily the fair case of $p = \frac{1}{2}$), with max. capital N and starting capital i, and for the expected duration of the game, and do the generalized versions of the theoretical problem **2** , and the Maple question **3**.

6. (Optional Challenge, 10 brownie points) Write a procedure

`EstimateProbSum(p1,p2,p3,p4,p5,p6,N1,N2,K1,K2)`

that inputs positive numbers $p_1, p_2, p_3, p_4, p_5, p_6$ (such that $p_1+p_2+p_3+p_4+p_5+p_6 = 1$) , positive integers `N1,N2` and (a not too large) integer K1, and a fairly large (more than 1000) integer K2, and uses simulation (by running the experiment `K2` times and recording each time whether it is success or failure), to estimate the probability of rolling a loaded die with prob. p_i of landing on i (for $1 \leq i \leq 6$) K times and getting a sum of the outcomes between `N1, and N2`.

What did you get for

`EstimateProbSum(1/6,1/6,1/6,1/6,1/6,1/6,100,330,360,2000);`

(run it several times, and see if you get close results). Do the same for

```
EstimateProbSum(0.1,0.1,0.1,0.1,0.1,0.5,100,430,470,2000);
```

7. (Optional Challenge, 20 brownie points) Write a procedure

```
ExactProbSum(p1,p2,p3,p4,p5,p6,N1,N2,K1)
```

that computes the above probability **exactly**. Hint: You need to know about *Probability Generating functions* and use the `coeff` command in Maple.