Homework for Lecture 11 of Dr. Z.'s Dynamical Models in Biology class

Email the answers (either as .pdf file or .txt file) to

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by 8:00pm Monday, Oct. 11,, 2021.

Subject: hw11

with an attachment hw11FirstLast.pdf and/or hw11FirstLast.txt

Also please indicate (EITHER way) whether it is OK to post

1. Read and understand the Maple code for the Maple procedure `SFPe(f,x)` . Use it prove that for $k < 3$ the difference equation

$$x_n = k\, x_{n-1}(1 - x_{n-1}) \quad ,$$

has one stable fixed point. What is it?

2. **(i)** By experimenting with the Maple procedure `Orb` for `f=k*x*(1-x)` (in M9.txt, also in M11.txt), starting at k=3.1, and incremetning it by 0.1 each time, i.e. $k = 3.1$ $k = 3.2$, and later on (once you get close to the bifurcation point), by $k = 0.01$ etc., estimate the value of k where the second bifurcation point happens, i.e. when the discrete Logistic equation switches from an ultimate (long-run) period of 2 to an ultimate (long-run) period of 4 (the second **period-doubling** event (as you know from (1), the first period-doubling, from period 1 to period 2, occurs exactly at $k = 3$.

ii (Optional, a little bit challenging, 5 brownie points),

by looking at the quartic function `Comp(k*x*(1-x),x)` (the composition of the $kx(1 - x)$ with itself, i.e. the function that maps today's value to the value at the day-after-tomorrow) and using `SFPe` find the EXACT value (up to the round-off error of Maple) of the number you found in (i).

3. Using `SFPe(f,x)` , for each of the three biological (single species) models given in section 3.1 of Professor Leah Edelstein-Keshet wonderful book

`https://sites.math.rutgers.edu/~zeilberg/Bio21/keshet/keshet3.pdf`

Do the following

- Find all the fixed points (expressed as expressions in the parameters of the model)

- For each of the above fixed point, expressed the conidition (in human language, not in Maple), in terms of the parameters of the model, for it to be a stable fixed point.

Here is a simple example to illustrate what I mean. For the difference equation

$$x_n = \frac{x_{n-1}}{x_{n-1}+c} \quad ,$$

with only one parameter (c) the answer would be

$x = 0$; It is a stable fixed point if and only if $c > 1$ or $c < -1$.

$x = 1 - c$. It is a stable fixed point if and only if $-1 < c < 1$.

4. Read and understand the Maple code for

`Orbk(k,z,f,INI,K1,K2)`.

(i) Use it to numerically find the *equilibrium point* (if it exist, I am telling you there is at most one) of the second-order differencec equation

$$x_n = \frac{x_{n-1} + a x_{n-2}}{b x_{n-1} + x_{n-2}}$$

using initial conditions $x_0 = 1.1, x_1 = 5.3$. Do it for **all** 16 choices of $a = 1, 2, 3, 4$; $b = 1, 2, 3, 4$.

(ii) By hand, find the nice explicit expression, in terms of a and b, for the only equilibrium point.

Hint: In the long run, (by the *definition* of **equilibrium**) all the values are the same so set $x_n = x_{n-1} = xn - 2 = z$ in the above recurrence, and solve for z (it is a simple equation). The root $z = 0$ should be ruled out, since if you plug-it-in you get division by 0.

Compare your answer to the numerical output of (i) (for those cases where there was a stable equilibrium, numerically).

5. (Optional Challenge, 20 brownie points). Find the condition phrased in terms of a and b for the equilibrium point you found in 4(ii) to be stable.

[This uses stuff that we will do later on. It is open until I get the first solution, so you are welcome to go back to it later.]

Conceptual comment; For **first-order** difference equation, "fixed-point" is the same as "equilibrium", but for higher-order recurrence where the underlying rule expresses x_n as a function of several variables, i.e. it is a function from R^k to R (where k is the order), the very name 'fixed-point' is **nonsense**.