Solutions to the Attendance quiz for Lecture 23

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1. For the following scenarios,

(i) Decide whether it is discrete-time or continuous-time dynamical model

(ii) Set up the appropriate difference or differential equation (as the case may be)

(iii) Write explicitly the **underlying transformation** (recall that for one quantity, like in the models below, it is simply a function of one variable)

(iv) Find **all** the fixed points (if it is discrete-time) or equilibrium points (if it is continuous time). Explain! Use the criterion for finding them.

(v) Find **all** the **stable** fixed points (if it is discrete-time) or all the **stable** equilibrium points (if it is continuous time). Explain! Use the criterion for finding them.

Call the quantity $x(t)$ or $x(n)$ for the continuous-time and discrete-time respectively.

a The population of a certain species is **decreasing** at a rate that is twice its current value.

b The population of a certain species changes from one generation to the next. The value at a given generation is one-half of its value at the previous generation.

c The population of a certain species changes from one generation to the next. The value at a given generation is twice its value at the previous generation times (1 minus its value at the previous generation).

d The population of a certain species scaled such that the maximum possible is 1 is **increasing** at a rate that is twice its current value times (1 minus its current value).

Before presenting the solution, here are **important reminders** about doing this problem where there is only **one** quantity.

Discrete Time:

• Format: $x(n) = f(x(n-1))$ where $f(x)$ is the **underlying function**.

To get the set of **fixed points** you use algebra and solve $x = f(x)$ (**not** $f(x) = 0$, this is for the continuous case).

Then you find $f'(x)$ and to decide whether a fixed point $x = a$ is **stable**, you compute $f'(a)$. If its **abslute value** is less than 1 it is stable, otherwise not.

Continuous Time:

- Format: $x'(t) = f(x(t))$ where $f(x)$ is the **underlying function**.

To get the set of **equilibrium points** you use algebra and solve $f(x) = 0$ (**not** $f(x) = x$, this is for the discrete case).

Then you find $f'(x)$ and to decide whether a fixed point $x = a$ is **stable**, you compute $f'(a)$. If it is **negative** than it is stable, otherwise not.

Note: For the one-quantity case, where the 'Jacobian' is a 1×1 matrix, that is always real, 'negative real part' is the same as 'negative'.

Sol. of a:

(i) Continuous time ; (ii) x'(t)=-2x(t) ; (iii) f(x)=-2x ; (iv) Solving $-2x = 0$; so there is only one equilibrium point, $x = 0$;

(v) $f'(x) = -2$, so $f'(0) = -2$ this is **negative** (in particular it has negative real part) so it is **stable**. So the set of stable equilibrium points is $\{0\}$.

Sol. of b:

(i) Discrete time; (ii) $x(n) = \frac{1}{2}x(n-1)$; (iii) $f(x) = \frac{1}{2}x$; (iv) Setting $x = f(x)$ gives $x = \frac{1}{2}x$. Solving this gives $x = 0$; so the set of fixed points is $\{0\}$;

(v) $f'(x) = \frac{1}{2}$, so $f'(0) = \frac{1}{2}$. Its **absolute value** is less than 1, so it is stable. So the set of stable fixed points is $\{0\}$.

Sol. of c:

(i) Discrete time; (ii) $x(n) = 2x(n-1)(1 - x(n-1))$; (iii) $f(x) = 2x(1-x)$; (iv) Setting $x = f(x)$ gives $x - 2x(1-x) = 0$. Simplifying: $x(1 - 2(1-x)) = x(2x-1)$. We have to solve $x(2x-1) = 0$, giving that the set of **fixed points** is $\{0, \frac{1}{2}\}$.

(v) $f(x) = 2x - 2x^2$, so $f'(x) = 2 - 4x$. When $x = 0$, $f'(0) = 2$. Since the absolute value of 2 is **not** less than 1, $x = 0$ is **not stable**. When $x = \frac{1}{2}$, $f'(\frac{1}{2}) = 0$. Since the absolute value of 0 is less than 1, $x = \frac{1}{2}$ is **stable**. So the set of **stable fixed points** is $\{\frac{1}{2}\}$

Sol. of d:

(i) Continuous time ; (ii) x'(t)=2x(t)(1-x(t)) ; (iii) f(x)=2x(1-x) ; (iv) Solving $2x(1-x) = 0$; gives $x = 0$ and $x = 1$, so the set of equilibrium points is $\{0, 1\}$.

(v) $f(x) = 2x - 2x^2$, So $f'(x) = 2 - 4x$. At $x = 0$, $f'(0) = 2$. Since this is **positive** (in fact **non-negative**, in particular the real part is not negative), it is **unstable**.

At $x = 1$, $f'(1) = -2$. Since this is **negative** it is **stable**. Hence the set of **stable** fixed points is $\{1\}$.