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AMLEH LADAS CONJECTURE 4

For this research project, Charles and I were investigating the third-order, rational difference equation as described below:

$$\text{(eq. i) } x_n = \frac{1+x_{n-1}}{x_{n-2}+x_{n-3}}.$$

Our goal was to show that for all positive initial conditions that the orbit ultimately ends up in a period six as shown:

$$\text{(eq. ii) } \{\phi, \psi, \frac{\psi}{\phi}, \frac{1}{\phi}, \frac{1}{\psi}, \frac{\phi}{\psi}\}$$

Additionally, we wanted to show the every positive solution of this difference equation (eq.i) converges to a positive solution of the difference equation below:

$$\text{(eq. iii) } x_n = \frac{x_{n-1}}{x_{n-2}}$$

We started off by analyzing various sample trajectories with different initial conditions and observed that the conjecture appears to be true. In order to show this more thoroughly, we developed a Maple procedure 'checkPeriodSixOrbit' which takes in variable 'k' and the underlying transformation 'F' and outputs a boolean representing whether the 1000-1005th values of k orbits, each with differing initial conditions, are of the format proposed by (eq. ii).

We then used this procedure to verify even further (of course, it is only a numerical verification and NOT a proof) by running it with k=1000 and got the output true. This means that for 1000 orbits with different initial conditions, the conjecture holds true.

The other major portion of the project was that we converted the underlying transformation of (eq. i) into a system and then composed that system with itself 6 times and called that 'Fcomp6.' The reasoning behind this composition was so that we can solve for the so-called fixed points of Fcomp6, where a fixed point would correspond to a value in the period 6 solution in (eq. ii). This is because when we compose the system with itself 6 times, it is the same as only looking at every 6th value of the original orbit/trajectory of (eq. i). If, as the conjecture states, the orbit eventually converges to a period 6 solution, then eventually every 6th value would be the same. Once we did this, the relevant fixed point we got is below:

$$\text{(eq. iv) } [z_1, z_2, z_2/z_1]$$

As a consequence, the fixed point (eq. iv) demonstrates that the relationship between the values in (eq. ii) match the recurrence relation of (eq. iii), therefore, using any values from (eq. ii) as initial conditions for the orbit of (eq. iii) will cause it to converge to an equivalent period 6 solution of (eq. i).

The remainder of our work involves experimentation with tweaking parameters of (eq. i) and analyzing the stability of some of them. We also have questions that were inspired by the beauty of these special recurrences and the spontaneous math that emerged when digging deeper.