Open Problems and Conjectures: SI and SIR Epidemic Models involving Discrete Time: An Investigation

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Basis of the Paper

Linda J.S. Allen's Paper "Open Problems and Conjectures: SI and SIR Epidemic Models" discusses an SIR Epidemic Model of the form:

$$x_{n+1} = x_n(1 - b - c) + y_n(1 - \exp(-ax_n))$$

$$y_{n+1} = (1 - y_n)b + y_n \exp(-ax_n),$$

where x_n represents the number of infected individuals and y_n represents the number of susceptible individuals (normalized to a population of size 1). In this model, $0 < b+c \leq 1$, $0 < a$, $0 < b$, $0 < c$, $0 < x_0+y_0 \leq 1$, $0 < x_0$, and $0 < y_0$. The birth/death rate is b, the recovery rate is c, and the contact rate is a.

Allen states that it has been proved that if $R_0 = a/(b+c) \leq 1$, then solutions approach [0, 1], the state with none infected and all susceptible (and for $R_0 > 1$, $[x_n, y_n]$ reaches a positive equilibrium). We investigated this phenomenon numerically and produced the following data:

a	b	c	R0	n = 10000	n = 10001	n = 10002
.3	.5	.5	.3000000000	[0., .9999999999]	[0., .9999999999]	[0., .9999999999]
.6	.5	.5	.6000000000	[0.1e-9, .9999999998]	[0.1e-9, .9999999998]	[0.1e-9, .9999999998]
.9	.5	.5	.9000000000	[0.5e-9, .9999999990]	[0.5e-9, .9999999990]	[0.5e-9, .9999999990]
1.2	.5	.5	1.200000000	[0.664898493e-1, .8670203...	[0.664898493e-1, .8670203...	[0.664898493e-1, .8670203...
1.5	.5	.5	1.500000000	[.1324566714, .7350866571]	[.1324566714, .7350866571]	[.1324566714, .7350866571]
1.8	.5	.5	1.800000000	[.1759239202, .6481521595]	[.1759239202, .6481521595]	[.1759239202, .6481521595]
2.1	.5	.5	2.100000000	[.2065468206, .5869063589]	[.2065468206, .5869063589]	[.2065468206, .5869063589]
2.4	.5	.5	2.400000000	[.2291534546, .5416930909]	[.2291534546, .5416930909]	[.2291534546, .5416930909]
2.7	.5	.5	2.700000000	[.2464264177, .5071471646]	[.2464264177, .5071471646]	[.2464264177, .5071471646]
3.0	.5	.5	3.000000000	[.2599754844, .4800490311]	[.2599754844, .4800490311]	[.2599754844, .4800490311]
3.3	.5	.5	3.300000000	[.2708251902, .4583496195]	[.2708251902, .4583496195]	[.2708251902, .4583496195]
3.6	.5	.5	3.600000000	[.2796586268, .4406827463]	[.2796586268, .4406827463]	[.2796586268, .4406827463]
3.9	.5	.5	3.900000000	[.2869487218, .4261025564]	[.2869487218, .4261025564]	[.2869487218, .4261025564]

Fig. 1: Holding b and c constant and incrementing a. Shows values of $[x_n, y_n]$ when n = 1000, 1001, and 1002.

a	b	c	R0	n = 10000	n = 10001	n = 10002
.5	.1	.1	2.500000000	[.2853911887, .4292176226]	[.2853911887, .4292176226]	[.2853911887, .4292176226]
.5	.2	.1	1.666666667	[.2419807546, .6370288682]	[.2419807546, .6370288682]	[.2419807546, .6370288682]
.5	.3	.1	1.250000000	[.1302503926, .8263328099]	[.1302503926, .8263328099]	[.1302503926, .8263328099]
.5	.4	.1	1.000000000	[0.1331361e-3, .9998335714]	[0.1331227e-3, .9998335881]	[0.1331093e-3, .9998336049]
.5	.5	.1	.8333333333	[0.9e-9, .9999999990]	[0.9e-9, .9999999990]	[0.9e-9, .9999999990]
.5	.6	.1	.7142857143	[0.3e-9, .9999999996]	[0.3e-9, .9999999996]	[0.3e-9, .9999999996]
.5	.7	.1	.6250000000	[0.3e-9, .9999999997]	[0.3e-9, .9999999997]	[0.3e-9, .9999999997]
.5	.8	.1	.5555555556	[0., 1.000000000]	[0., 1.000000000]	[0., 1.000000000]
.5	.9	.1	.5000000000	[0., 1.000000000]	[0., 1.000000000]	[0., 1.000000000]

Fig. 2: Holding a and c constant and incrementing c. Shows values of $[x_n, y_n]$ when n = 1000, 1001, and 1002.

a	b	c	R0	n = 10000	n = 10001	n = 10002
.5	.1	.1	2.500000000	[.2853911887, .4292176226]	[.2853911887, .4292176226]	[.2853911887, .4292176226]
.5	.1	.2	1.666666667	[.1269202075, .6192393774]	[.1269202075, .6192393774]	[.1269202075, .6192393774]
.5	.1	.3	1.250000000	[0.476100524e-1, .8095597...	[0.476100524e-1, .8095597...	[0.476100524e-1, .8095597...
.5	.1	.4	1.000000000	[0.377424e-4, .9998111356]	[0.377386e-4, .9998111546]	[0.377348e-4, .9998111736]
.5	.1	.5	.8333333333	[0.9e-9, .9999999946]	[0.9e-9, .9999999946]	[0.9e-9, .9999999946]
.5	.1	.6	.7142857143	[0.3e-9, .9999999976]	[0.3e-9, .9999999976]	[0.3e-9, .9999999976]
.5	.1	.7	.6250000000	[0.3e-9, .9999999976]	[0.3e-9, .9999999976]	[0.3e-9, .9999999976]
.5	.1	.8	.5555555556	[0., .9999999996]	[0., .9999999996]	[0., .9999999996]
.5	.1	.9	.5000000000	[0., .9999999996]	[0., .9999999996]	[0., .9999999996]

Fig. 3: Holding a and b constant and incrementing c. Shows values of $[x_n, y_n]$ when n = 1000, 1001, and 1002.

Clearly, it holds that for $R_0 \leq 1$, solutions approach $[x_n, y_n] = [0,1]$.

Conjecture 1

Conjecture 1 discussed the case where the reproductive number of the disease was greater than one. This implies that the disease will not be fully eradicated, but will reach an equilibrium with some infected and some vulnerable. The reproductive number, $R_0 = a/(b+c)$. In the specific case where $R_0 > 1$, a set of difference equations were proposed:

$$(b+c)x^* = y^*(1-\exp(-ax^*)) \quad \text{and} \quad y^* = 1 - x^*(1+c/b). \qquad (3)$$

Dr. Linda Allen and Dr. Gerry Ladas proposed the following conjecture based on the above equation:

CONJECTURE 1 *Prove if $R_0 = a/(b+c) > 1$, then the solution (x_n, y_n) to the first order system* (1) *and* (2) *satisfies*

$$\lim_{n \to \infty} (x_n, y_n) = (x^*, y^*),$$

where x^* and y^* are the positive solutions of (3).

In order to fully investigate this conjecture, it was hypothesized that this should hold for any value of a, b, or c. As such, methods were written that held two of the variables constant, and incremented the remaining variable. Then the long term solution was determined and compared to the solution to equation 3 given above.

The data tables for each situation is given as follows:

aROver1 := ["Table A" "Table A"]

a	R	Numeric Equilibrium Solution at n = 1010	Solution According to eq3
1.	1.2500...	[0.651251964e-1, .8263328098]	[0.6512519632e-1, .8263328096]
1.2	1.5000...	[.1083892630, .7109619654]	[.1083892630, .7109619653]
1.4	1.7500...	[.1391625190, .6288999495]	[.1391625190, .6288999493]
1.6	2.0000...	[.1621301995, .5676528015]	[.1621301995, .5676528013]
1.8	2.2500...	[.1798952765, .5202792627]	[.1798952765, .5202792626]
2.0	2.5000...	[.1940195884, .4826144311]	[.1940195884, .4826144309]
2.2	2.7500...	[.2054970777, .4520077926]	[.2054970778, .4520077927]
2.4	3.0000...	[.2149904132, .4266922313]	[.2149904132, .4266922314]
2.6	3.2500...	[.2229583990, .4054442695]	[.2229583990, .4054442693]
2.8	3.5000...	[.2297287810, .3873899172]	[.2297287810, .3873899173]
3.0	3.7500...	[.2355419314, .3718881829]	[.2355419314, .3718881829]
3.2	4.0000...	[.2405781504, .3584582658]	[.2405781504, .3584582655]
3.4	4.2500...	[.2449753321, .3467324476]	[.2449753321, .3467324477]
3.6	4.5000...	[.2488407433, .3364246845]	[.2488407432, .3364246845]
3.8	4.7500...	[.2522590804, .3273091190]	[.2522590804, .3273091188]
4.0	5.0000...	[.2552981110, .3192050372]	[.2552981110, .3192050372]
4.2	5.2500...	[.2580127035, .3119661240]	[.2580127035, .3119661239]
4.4	5.5000...	[.2604477566, .3054726491]	[.2604477565, .3054726490]
4.6	5.7500...	[.2626403659, .2996256910]	[.2626403659, .2996256908]
4.8	6.0000...	[.2646214484, .2943428042]	[.2646214484, .2943428042]
5.0	6.2500...	[.2664169772, .2895547276]	[.2664169772, .2895547274]

Fig.4: Incrementing a from 1 to 5 by 0.2 each time; b and c are held constant.

$bROver1 := \lfloor$ "Table B" "Table B" $\rfloor$

b	R	Numeric Equilibrium Solution at n = 1010	Solution According to eq3
.1	9.0909...	[.7659514682, .1574533845]	[.7659514686, .1574533850]
.15	6.2500...	[.7264746429, .2250937143]	[.7264746431, .2250937140]
.20	4.7619...	[.6770889812, .2890565695]	[.6770889814, .2890565697]
.25	3.8461...	[.6252849838, .3497036169]	[.6252849838, .3497036168]
.30	3.2258...	[.5735331840, .4073490433]	[.5735331839, .4073490434]
.35	2.7777...	[.5227970737, .4622658671]	[.5227970736, .4622658668]
.40	2.4390...	[.4734708652, .5146923632]	[.4734708651, .5146923632]
.45	2.1739...	[.4257024179, .5648375284]	[.4257024178, .5648375285]
.50	1.9607...	[.3795239317, .6128855897]	[.3795239318, .6128855897]
.55	1.7857...	[.3349110028, .6589997062]	[.3349110029, .6589997063]
.60	1.6393...	[.2918114549, .7033250209]	[.2918114549, .7033250208]
.65	1.5151...	[.2501601949, .7459911867]	[.2501601948, .7459911868]
.70	1.4084...	[.2098871419, .7871144703]	[.2098871420, .7871144704]
.75	1.3157...	[.1709215330, .8267995132]	[.1709215330, .8267995133]
.80	1.2345...	[.1331942543, .8651408175]	[.1331942543, .8651408175]
.85	1.1627...	[0.966390701e-1, .9022239997]	[0.9663907014e-1, .9022239996]
.90	1.0989...	[0.611932135e-1, .9381268619]	[0.6119321357e-1, .9381268619]

Fig. 5: Incrementing b from 0.1 to 0.9 by 0.05; a and c held constant

$cROver1 := \lfloor$ "Table C" "Table C" $\rfloor$

c	R	Numeric Equilibrium Solution at n = 1010	Solution According to eq3
0.5e-1	6.2500...	[.5448268880, .2075245260]	[.5448268884, .2075245268]
.10	4.7619...	[.3909121318, .2537132030]	[.3909121320, .2537132030]
.15	3.8461...	[.2959952348, .3003748996]	[.2959952345, .3003748995]
.20	3.2258...	[.2316089330, .3472839167]	[.2316089328, .3472839161]
.25	2.7777...	[.1850633087, .3943382624]	[.1850633087, .3943382624]
.30	2.4390...	[.1498454358, .4414851939]	[.1498454358, .4414851939]
.35	2.1739...	[.1222686548, .4886947166]	[.1222686547, .4886947163]
.40	1.9607...	[.1000895345, .5359485218]	[.1000895345, .5359485219]
.45	1.7857...	[0.818645923e-1, .5832348027]	[0.8186459234e-1, .5832348028]
.50	1.6393...	[0.666229234e-1, .6305456060]	[0.6662292352e-1, .6305456066]
.55	1.5151...	[0.536874376e-1, .6778753743]	[0.5368743755e-1, .6778753744]
.60	1.4084...	[0.425715309e-1, .7252201186]	[0.4257153086e-1, .7252201187]
.65	1.3157...	[0.329165029e-1, .7725768895]	[0.3291650286e-1, .7725768891]
.70	1.2345...	[0.244521215e-1, .8199434694]	[0.2445212152e-1, .8199434689]
.75	1.1627...	[0.169709355e-1, .8673181405]	[0.1697093548e-1, .8673181406]
.80	1.0989...	[0.103110400e-1, .9146995783]	[0.1031103995e-1, .9146995782]

Fig. 6: Incrementing c from 0.05 to 0.8 by 0.05; a and b held constant

Thorough examination of each table shows that the long term values using the OrbF function do match the values determined by equation 3 proposed by Dr. Allen when $R_0 > 1$.

In addition, the functions were run for cases where $R_0 \leq 1$, in order to see how the solutions to equation 3 would compare to the long term behavior of the discrete dynamical model. This information is given below:

$aLE1 := \left[\text{"Table A"} \quad \text{"Table A1"} \right]$

a	R	Numeric Equilibrium Solution at n = 1010	Solution According to eq3
0.5e-1	0.5263...	[0., 1.000000000]	[0., 1.]
.10	.10526...	[0., 1.000000000]	[0., 1.]
.15	.15789...	[0., 1.000000000]	[0., 1.]
.20	.21052...	[0., 1.000000000]	[0., 1.]
.25	.26315...	[0., 1.000000000]	[0., 1.]
.30	.31578...	[0., 1.000000000]	[0., 1.]
.35	.36842...	[0., 1.000000000]	[0., 1.]
.40	.42105...	[0., 1.000000000]	[0., 1.]
.45	.47368...	[0., 1.000000000]	[0., 1.]
.50	.52631...	[0., 1.000000000]	[0., 1.]
.55	.57894...	[0.1e-9, .9999999999]	[0.1052631579e-9, .9999999999]
.60	.63157...	[0.1e-9, .9999999999]	[0.1052631579e-9, .9999999999]
.65	.68421...	[0.1e-9, .9999999999]	[0.1052631579e-9, .9999999999]
.70	.73684...	[0.1e-9, .9999999999]	[0.1052631579e-9, .9999999999]
.75	.78947...	[0.2e-9, .9999999997]	[0.2105263157e-9, .9999999997]
.80	.84210...	[0.2e-9, .9999999997]	[0.2105263157e-9, .9999999997]

Fig 7: Incrementing a, $R_0 \leq 1$

$bLE1 := \left[\text{"Table B"} \quad \text{"Table B1"} \right]$

b	R	Numeric Equilibrium Solution at n = 1010	Solution According to eq3
.5	.98039...	[0.4416e-6, .9999995494]	[0.3915684678e-6, .9999995926]
.55	.89285...	[0.13e-8, .9999999987]	[0.1249999998e-8, .9999999987]
.60	.81967...	[0.9e-9, .9999999991]	[0.8196721305e-9, .9999999991]
.65	.75757...	[0.5e-9, .9999999995]	[0.4545454542e-9, .9999999995]
.70	.70422...	[0.3e-9, .9999999997]	[0.2816901407e-9, .9999999997]
.75	.65789...	[0.3e-9, .9999999997]	[0.2631578946e-9, .9999999997]
.80	.61728...	[0.3e-9, .9999999997]	[0.2469135801e-9, .9999999997]
.85	.58139...	[0., 1.000000000]	[0., 1.]
.90	.54945...	[0., 1.000000000]	[0., 1.]

Fig. 8: Incrementing b, $R_0 \leq 1$

$cLE1 := [$ "Table C" "Table C1" $]$

c	R	Numeric Equilibrium Solution at n = 1010	Solution According to eq3
0.5e-1	.66666...	[0.16e-8, .9999999976]	[0.1333333330e-8, .9999999976]
.10	.50000...	[0.7e-9, .9999999986]	[0.4999999993e-9, .9999999986]
.15	.40000...	[0.1e-9, .9999999996]	[0., .9999999998]
.20	.33333...	[0.1e-9, .9999999996]	[0., .9999999997]
.25	.28571...	[0.1e-9, .9999999996]	[0., .9999999996]
.30	.25000...	[0.1e-9, .9999999996]	[0., .9999999996]
.35	.22222...	[0.1e-9, .9999999996]	[0., .9999999996]
.40	.20000...	[0., .9999999996]	[0., 1.]
.45	.18181...	[0., .9999999996]	[0., 1.]
.50	.16666...	[0., .9999999996]	[0., 1.]
.55	.15384...	[0., .9999999996]	[0., 1.]
.60	.14285...	[0., .9999999996]	[0., 1.]
.65	.13333...	[0., .9999999996]	[0., 1.]
.70	.12500...	[0., .9999999996]	[0., 1.]
.75	.11764...	[0., .9999999996]	[0., 1.]
.80	.11111...	[0., .9999999996]	[0., 1.]

Fig. 9: Incrementing c, $R_0 \leq 1$

It seems that overall the long term behavior and the value of equation 3 are about the same, regardless of which variable you fix and which is incremented. However, there are some values of a, b, and c (marked in red) such that we see the long term behavior does not quite match the value of equation 3. This could be due to Maple's approximation, so it cannot quite be concluded whether or not Conjecture 1 holds for $R_0 \leq 1$.

<u>Conjecture 2</u>

For conjecture 2, we investigated the case where there was no recovery in the population (i.e. c = 0) called the SI Model. For the scenario c = 0 and $x_0 + y_0 = 1$, a unique second order difference equation arises:

$$x_{n+1} = x_n(1-b) + (1-x_n)(1-\exp(-ax_{n-1})), \quad n = 1, 2, \ldots, \quad (4)$$

In this case the conjecture was:

CONJECTURE 2 *Prove if $R_0 = a/b > 1$, then the solution of (4) satisfies*

$$\lim_{n \to \infty} x_n = x^*,$$

where x^* is the positive solution of

$$bx^* = (1 - x^*)(1 - exp(-ax^*)).$$

We first normalized the equation to x(n) rather than x(n+1) and created a function which would form the first order system of the equation:

```
AllenSI :=proc(a, b, x)
 ToSys(2, x, x[1]·(1 − b) + (1 − x[1])·(1 − exp(-a·x[2])))[1]:
 end:
```

The output to this function, taking a and b as general parameters, was in the form:

$$\left[x_1 (1 - b) + \left(1 - x_1\right) \left(1 - e^{-a x_2}\right), x_1\right]$$

We also created a function that would output the variables of the system after converting it into a first order system:

```
AllenSIVar :=proc(a, b, x)
 ToSys(2, x, x[1]·(1 − b) + (1 − x[1])·(1 − exp(-a·x[2])))[2]:
 end:
```

When investigating the long term behavior of equation 4, we found that the conjecture does hold for R0 = a/b > 1 (it differs from a/(b+c) because in this case c = 0). The subsequent charts exhibit the long term behavior of equation 4, the R_0 value, and the subsequent solution for the SI Model Equation in the conjecture.

In the case below we incremented the parameter a, for a constant b = 0.5 with initial conditions $x_1(0) = 0.5$ and $x_2(0) = 0.5$:

a	b	c	R0	n=1000	n=1001	n=1002	SI Model Equation
0	.5	0	0.	0.1e-9	0.1e-9	0.1e-9	0.
.1	.5	0	.2000000000	0.1e-9	0.1e-9	0.1e-9	0.
.2	.5	0	.4000000000	0.1e-9	0.1e-9	0.1e-9	0.
.3	.5	0	.6000000000	0.5e-9	0.5e-9	0.5e-9	0.
.4	.5	0	.8000000000	0.9e-9	0.9e-9	0.9e-9	0.1e-8
.5	.5	0	1.000000000	0.23752800e-2	0.23729293e-2	0.23705832e-2	0.2368231e-2
.6	.5	0	1.200000000	.1329797001	.1329797001	.1329797001	.132979700
.7	.5	0	1.400000000	.2273657171	.2273657171	.2273657171	.227365717
.8	.5	0	1.600000000	.2976398435	.2976398436	.2976398435	.297639843
.9	.5	0	1.800000000	.3518478408	.3518478407	.3518478408	.3518478406
1.0	.5	0	2.000000000	.3948174229	.3948174228	.3948174229	.3948174228
1.1	.5	0	2.200000000	.4296209744	.4296209744	.4296209744	.4296209742
1.2	.5	0	2.400000000	.4583069093	.4583069092	.4583069093	.4583069091
1.3	.5	0	2.600000000	.4822934828	.4822934828	.4822934828	.4822934826
1.4	.5	0	2.800000000	.5025938260	.5025938261	.5025938260	.5025938261
1.5	.5	0	3.000000000	.5199509688	.5199509689	.5199509688	.5199509690
1.6	.5	0	3.200000000	.5349222292	.5349222293	.5349222292	.5349222294
1.7	.5	0	3.400000000	.5479338200	.5479338201	.5479338200	.5479338201
1.8	.5	0	3.600000000	.5593172536	.5593172538	.5593172536	.5593172537
1.9	.5	0	3.800000000	.5693342518	.5693342520	.5693342518	.5693342519
2.0	.5	0	4.000000000	.5781941830	.5781941830	.5781941830	.5781941830

Fig. 10: Incrementing a

We see that for $R_0 \leq 1$, the solution of equation 4 *does not* satisfy the SI Equation given in the conjecture. However for $R_0 > 1$, the solution *does* satisfy the SI Equation.

When we increment b instead, for b < 1, as that is the original boundary for the SIR model, and hold a at 0.1, we get a slightly different result:

a	b	c	R0	n=1000	n=1001	n=1002	SI Model Equation
.1	.1	0	1.000000000	0.102073328e-1	0.101973840e-1	0.101874546e-1	0.10097989e-1
.1	.2	0	.5000000000	0.7e-9	0.7e-9	0.7e-9	0.
.1	.3	0	.3333333333	0.1e-9	0.1e-9	0.1e-9	0.
.1	.4	0	.2500000000	0.1e-9	0.1e-9	0.1e-9	0.
.1	.5	0	.2000000000	0.1e-9	0.1e-9	0.1e-9	0.
.1	.6	0	.1666666667	0.	0.	0.	0.
.1	.7	0	.1428571429	0.	0.	0.	0.
.1	.8	0	.1250000000	0.	0.	0.	0.
.1	.9	0	.1111111111	0.	0.	0.	0.
.1	1.0	0	.1000000000	0.	0.	0.	0.

Fig. 11: Incrementing b, $R_0 \leq 1$

For R_0 values that are very close to 0, the long term behavior approaches 0, and therefore the SI Model Equation is satisfied even if $R_0 < 1$. However, this may be attributed to the fact that Maple approximates those very small values to 0 and therefore it seems to be equal when it really is not.

When we test larger values for R_0 by incrementing a and keeping b = 0.9, we find the same result that the SI Model Equation is satisfied at $R_0 > 1$:

a	b	c	R0	n=1000	n=1001	n=1002	SI Model Equation
0	.9	0	0.	0.	0.	0.	0.
.5	.9	0	.5555555556	0.	0.	0.	0.
1.0	.9	0	1.111111111	0.687212646e-1	0.687212645e-1	0.687212646e-1	0.687212645e-1
1.5	.9	0	1.666666667	.2702125770	.2702125736	.2702125770	.2702125760
2.0	.9	0	2.222222222	.5420292667	.2131629584	.5420292667	.3367509387
2.5	.9	0	2.777777778	.6899952057	.1826337978	.6899952057	.2830773688
3.0	.9	0	3.333333333	.7583358437	.1741938643	.7583358437	.2409133502
3.5	.9	0	3.888888889	.7909118185	.1747592563	.7909118185	.2177361987
4.0	.9	0	4.444444444	.8030451120	.1822501714	.8030451120	.2100264089
4.5	.9	0	5.000000000	.7997954016	.1980811183	.7997954016	.2163658052
5.0	.9	0	5.555555556	.2297469551	.7774351076	.2297469551	.5845042642
5.5	.9	0	6.111111111	.4735856964	.5478979375	.4734707833	.5416664358
6.0	.9	0	6.666666667	.5146697375	.5146699914	.5146697404	.5146699817
6.5	.9	0	7.222222222	.5175301477	.5175301512	.5175301477	.5175301510
7.0	.9	0	7.777777778	.5196633045	.5196633070	.5196633045	.5196633069

Fig. 12: Incrementing a, R_0 at large values

Here we see that at larger values of R_0, the conjecture holds. However, like in Figure 11, for very small values of R_0, the conjecture holds at x = 0. However, that again could be attributed to Maple approximating the smaller fixed points at 0, rather than the actual values.

<u>Further Investigation: Higher Order Terms</u>

We investigated a similar SIR Model of the form:

$$x_{n+1} = x_n^{\alpha}(1-b-c) + y_n^{\beta}\left(1-e^{-a x_n}\right)$$

$$y_{n+1} = \left(1-y_n^{\beta}\right) b + y_n^{\beta} e^{-a x_n}$$

introducing the two new parameters α and β. While it was immediately evident that $R_0 = a/(b+c) \leq 1$ no longer implied $[x_n, y_n]$ approaches [0,1] as n approaches infinity, we hypothesized that a new formula of the form:

$R_0 = f(a, b, c, \alpha, \beta)$ might exist such that if $R_0 = f(a, b, c, \alpha, \beta) \leq 1$, then the value of x_n approaches 0 as n approaches infinity (and for $R_0 > 1$, x_n reaches a positive equilibrium).

It should be noted that, unlike in the original case (with $\alpha = \beta = 0$), we have defined R_0 such that, for $R_0 \leq 1$, it is true that x_n approaches 0 as n approaches infinity, but it is not necessarily true that y_n approaches 1 as n approaches infinity.

To explore what the function f(a, b, c, α, β) might be, we defined a_0 to be the last value of a such that, for given b, c, α, and β, x_n approaches the infection-free state. In other words, it is the cutoff point after which the number of infected individuals will be nonzero in the long run. The plot below illustrates this concept for the original model, where $\alpha = \beta = 1$.

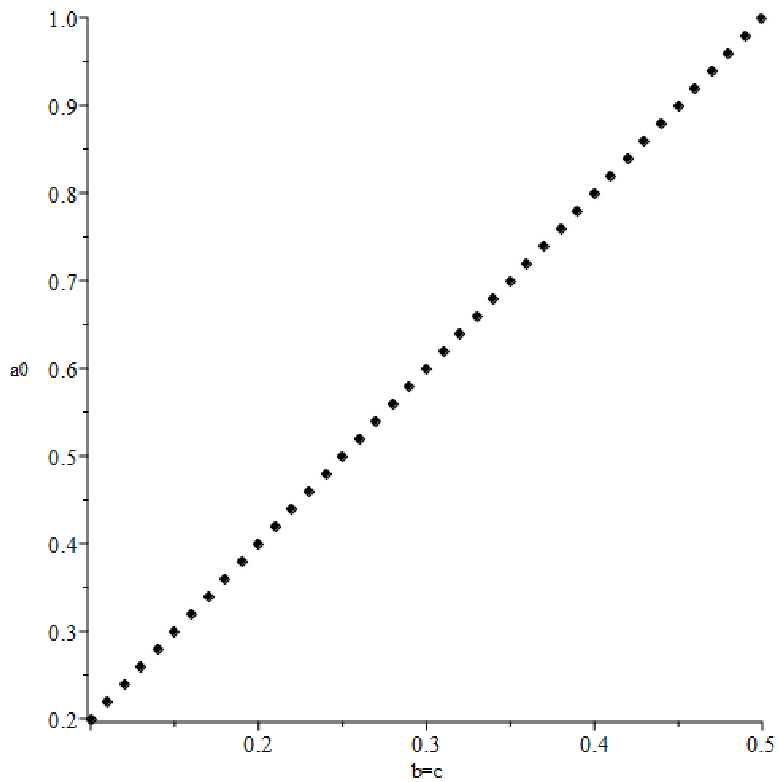

Fig.13: For any b, c, if a is any greater than b+c, then x_n is nonzero in the long run. Thus, for any b, c, a_0 = b+c.

We made similar plots, showing values of a_0 as b, c, α, β are incremented separately.

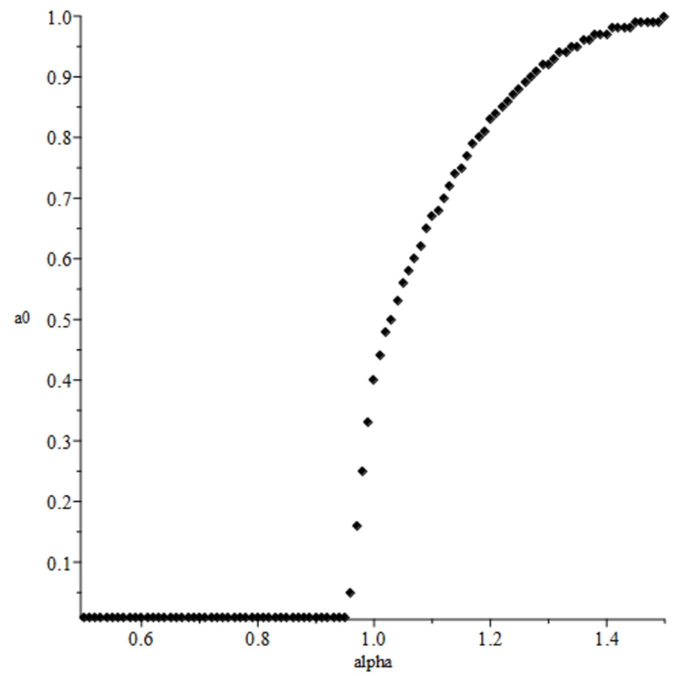

Fig: 14: Holding β constant at 1.1 and b, c constant at 0.2 and incrementing α from 0.5 to 1.5. We believe that for any α < 1, $a_0 = 0$, meaning there are always infected individuals in the long run (the points near α = 1 which seem to suggest otherwise are likely due to rounding error, but this remains unconfirmed).

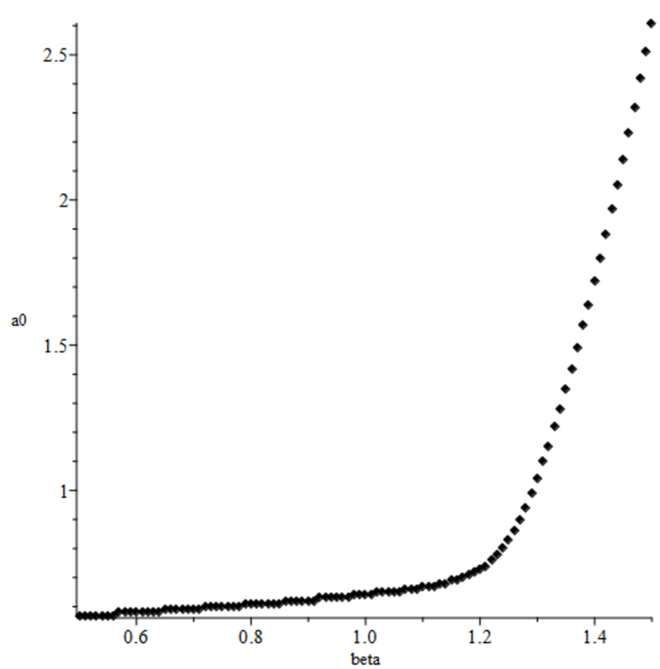

Fig. 15: Holding α constant at 1.1 and b, c constant at 0.2 and incrementing β from 0.5 to 1.5.

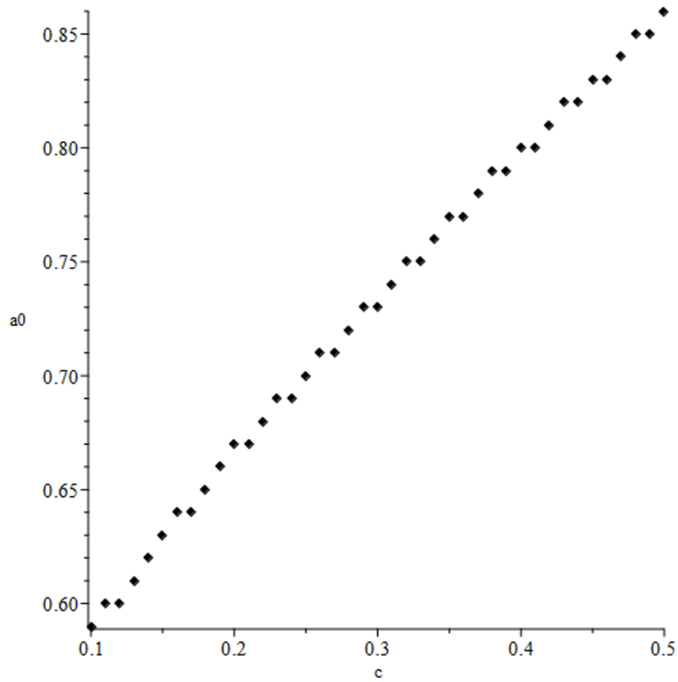

Fig. 16: Holding α, β constant at 1.1 and b constant at 0.2 and incrementing c from 0.1 to 0.5.

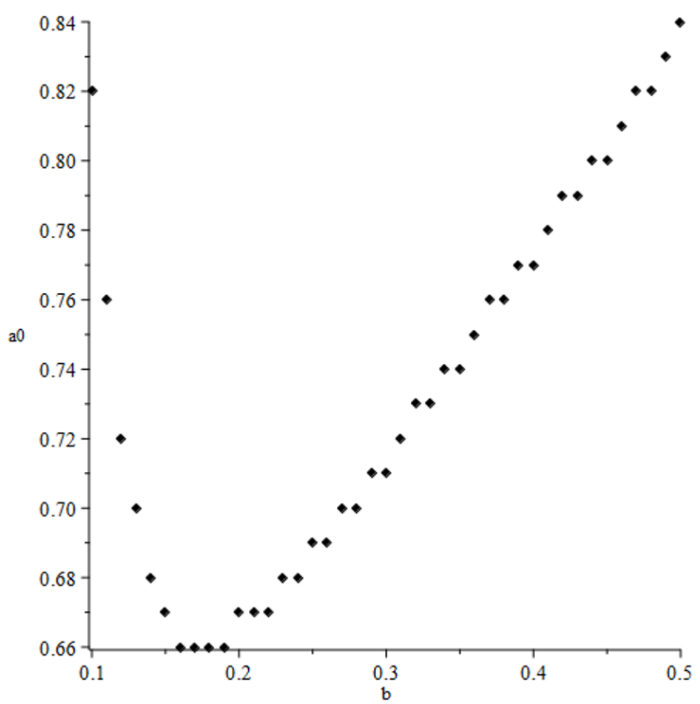

Fig. 17: Holding α, β constant at 1.1 and c constant at 0.2 and incrementing b from 0.1 to 0.5. There appears to be an asymptote as b approaches 0.

Though it remains unclear exactly what form $R_0 = f(a, b, c, \alpha, \beta)$ actually takes, these plots give some clues. The final plot is perhaps the most interesting, as it appears to show that a_0 approaches infinity as b approaches 0. This does not hold true as c approaches 0, implying that b and c are not interchangeable as they were in the original case $R_0 = a/(b+c)$.

Based on these plots, we make the following conjectures:

1. For $\alpha < 1$, x_n is always nonzero as n approaches infinity.
2. For $b > 0.2$, a_0 increases (or remains constant) as any of b, c, α, β increase.
3. As b approaches 0, a_0 approaches infinity.