OK to post

Anne Somalwar, 9/27/2021, hw7

2. (a)

$x_N(i)$ becomes $x_N(i-1)$ if you lose a round (you have yet another dollar to make up). $x_N(i)$ becomes $x_N(i+1)$ if you win a round (you have one less dollar you will need to win). There is an equal chance of winning or losing each round, so $x_N(i) = \frac{1}{2}(x_N(i-1) + x_N(i+1))$.

The boundary conditions are true because, if $i=0$, you automatically lose and your chances of winning are 0. If $i=N$, you automatically win and your chances of winning are 1.

(b) $\frac{1}{2}\left(\frac{i-1}{N}+\frac{i+1}{N}\right)=\frac{i}{N}$. ✓

$y_N(0)=\frac{0}{N}=0$. ✓

$y_N(N)=\frac{N}{N}=1$. ✓

We have this solution because, for any nonzero c,

$\frac{1}{2}(c(i-1)+c(i+1))=ci$. Since

$x_N(N) = 1$, $c = \frac{1}{N}$ and $c_i = \frac{i}{N}$.

(c) Given i dollars, if you lose a round, you have $i-1$ dollars. If you win, you have $i+1$. There is an equal chance of winning or losing, so

$E_N(i) = \frac{1}{2}(E_N(i-1) + E_N(i+1)) + 1$ (the 1 is added for the current round that has been played).

The boundary conditions are true because, if $i=0$, you automatically lose and there are no expected rounds. If $i = N$, you automatically win and there are no rounds.

(d) $\frac{1}{2}(z_N(i-1) + z_N(i+1)) + 1$

$= \frac{1}{2}((i-1)(N-i+1) + (i+1)(N-i-1)) + 1$

$= \frac{1}{2}(Ni - i^2 + \cancel{i} - \cancel{N} + \cancel{i} - 1 + Ni - i^2 - \cancel{i} + \cancel{N} - \cancel{i} - 1) + 1$

$= \frac{1}{2}(2(Ni - i^2 - 1)) + 1$

$= Ni - i^2$

$= i(N-i) = z_N(i)$. ✓

We have this formula because

$\frac{1}{2}((i-1)(c-(i-1)) + (i+1)(c-(i+1))) + 1$

$= i(c-i)$ for any c, and

since $E_N(N) = 0$, $c = N$.

5)(a) Recurrence

$$x_N(i) = p\, x_N(i+1) + (p-1)\, x_N(i-1)$$

This is the case because there is probability p that the player will win a round, in which case the probability of winning becomes $x_N(i+1)$. There is probability $p-1$ that the player will lose a round, in which case the probability of winning becomes $x_N(i-1)$.

(b) Explicit Form

$$x_N(i) = \begin{cases} \dfrac{1-\left(\frac{(1-p)}{p}\right)^i}{1-\left(\frac{(1-p)}{p}\right)^N} & \text{if } p \neq 0.5 \\ i/N & \text{if } p = 0.5 \end{cases}$$

The boundary conditions remain the same.

(c)

Recurrence

$$E_N(i) = pE_N(i+1) + (p-1)E_N(i-1) + 1$$

This is the case because there is probability p that the player will win a round, in which case the expected # of rounds becomes $E_N(i+1)$. There is probability $p-1$ that the player will lose a round, in which case the expected # of rounds becomes $E_N(i-1)$.

$M=0$, $k=i$

<u>Explicit Form</u>

$$E_N(i) = \begin{cases} \dfrac{i}{1-2p} - \dfrac{N}{1-2p}\dfrac{\left(\left(\frac{1-p}{p}\right)^i - 1\right)}{\left(\left(\frac{1-p}{p}\right)^N - 1\right)} & \text{when } p \neq 0.5 \\ i(N-i) & \text{when } p = 0.5 \end{cases}$$

The boundaries are the same.