#Please do not post homework
#Julian Herman, 9/27/2021, Assignment 7

2.) a.) $X_N(i) = \frac{1}{2}\left(X_N(i-1) + X_N(i+1)\right)$

$\Rightarrow$ This linear homogenous recurrence is true because for a fixed N, if $X_N(i)$ is the probability of winning with i dollars, there are two events that could occur: you lose a dollar $(i-1)$
you gain a dollar $(i+1)$

Both of these events have a 50% chance or $\frac{1}{2}$ probability of occuring (for a fair coin).
Therefore, by the conservation of probability, the sum of the probability of winning from each of these states $[X_N(i+1) + X_N(i-1)]$ multiplied by the odds of entering one of these states $(\frac{1}{2})$ must be equivalent to the probability of winning from the initial state
$\hookrightarrow X_N(i)$

Alternatively, it could be thought of as there being two states from which one could arrive at i dollars, you either lost a dollar (came from $i+1$) or gained a dollar previously (came from $i-1$).
Then, by the Law of Total Probability:

$$P(X_N(i)) = \underbrace{P(X_N(i) \mid X_N(i-1))}_{\text{the probability of going from } (i-1) \text{ to } i \,=\, \frac{1}{2}} \cdot P(X_N(i-1)) + \underbrace{P(X_N(i) \mid X_N(i+1))}_{\text{the probability of going from } (i+1) \text{ to } i \,=\, \frac{1}{2}} \cdot P(X_N(i+1))$$

Boundary conditions:

$X_N(0) = 0$ is true because the probability of winning when you have $i = \varnothing$ dollars is $\varnothing$ because you cannot gamble with $\varnothing$ dollars.

$X_N(N) = 1$ is true because the probability of winning, which is defined as leaving with N dollars, is 100% (or $1/1$) if you already have N dollars.

b) $y_N(i) = i/N$

boundary conditions:

$$y_N(0) = \frac{0}{N} = 0 \checkmark$$

$$y_N(N) = \frac{N}{N} = 1 \checkmark$$

$$y_N(i) \stackrel{?}{=} \frac{1}{2}\left(y_N(i-1) + y_N(i+1)\right)$$

$$\frac{i}{N} \stackrel{?}{=} \frac{1}{2}\left(\frac{(i-1)}{N} + \frac{(i+1)}{N}\right)$$

$$= \frac{1}{2}\left(\frac{i}{N} - \cancel{\frac{1}{N}} + \frac{i}{N} + \cancel{\frac{1}{N}}\right)$$

$$= \frac{1}{\cancel{2}}\left(\cancel{2} \cdot \frac{i}{N}\right)$$

$$\frac{i}{N} = \frac{i}{N} \checkmark$$

LHS = RHS

We have the beautiful explicit formula $X_N(i) = \frac{i}{N}$ because the chance to gain a dollar at each round is equivalent to the chance to lose a dollar at each round (both $= \frac{1}{2}$). This means that the probability of obtaining N dollars is based solely on how much money you have (i dollars) in relation to the amount you must obtain to win (N). So if you're starting with $i << N$ and the probability to win and lose are the same, it makes sense that you're more likely to lose because you're starting closer to 0 (same vice versa). This is why the overall probability to win is the fraction of i/N.

c) $E_N(i) = \frac{1}{2}\left(E_N(i-1) + E_N(i+1)\right) + 1$ is true because there is $\frac{1}{2}$ probability of losing a dollar $(i-1)$ and $\frac{1}{2}$ probability of gaining a dollar $(i+1)$ when you have i dollars. If <u>either</u> <u>one</u> of these events occur (and only <u>one</u> can occur <u>per</u> <u>round</u> and <u>rounds</u> are <u>independent</u>), then 1 must be added because a round is used when switching states (from i to either $i+1$ or $i-1$).

Boundary conditions:

$E_N(0) = 0$ is true because the game is over if you have 0 dollars, therefore the expected number of rounds will always be 0. "You've lost before starting"

$E_N(N) = 0$ is true for the same reasoning:
If you already have N dollars, the game is over, therefore the expected number of rounds is 0.
"You've won before starting"

d) $Z_N(i) = i(N-i)$

Boundary:

$$Z_N(0) \overset{?}{=} 0$$
$$0 \cdot (N-0) = 0$$
$$0 \cdot N = 0$$
$$0 = 0 \checkmark$$

$$Z_N(N) \overset{?}{=} 0$$
$$N(N-N) = 0$$
$$N \cdot 0 = 0$$
$$0 = 0 \checkmark$$

$$Z_N(i) \overset{?}{=} \frac{1}{2}\left(Z_N(i-1) + Z_N(i+1)\right) + 1$$

$$i(N-i) \overset{?}{=} \frac{1}{2}\left((i-1)\left(N-(i-1)\right) + (i+1)\left(N-(i+1)\right)\right) + 1$$

$$= \frac{1}{2}\left(iN \cancel{- N} - (i-1)^2 + iN \cancel{+ N} - (i+1)^2\right) + 1$$

$$= \frac{1}{2}\left(2iN - i^2 \cancel{+2i} - 1 - i^2 \cancel{-2i} - 1\right) + 1$$

$$= \frac{1}{\cancel{2}}\left(\cancel{2}iN - \cancel{2}i^2 - \cancel{2}\right) + 1$$

$$= iN - i^2 \cancel{-1} \cancel{+1}$$

$$i(N-i) = i(N-i) \checkmark \quad \text{LHS} = \text{RHS}$$

⇒ NOT sure why we have this formula...