09/09 HW 2

1.

$a_n = 4a_{n-1} - 6a_{n-2} + 4a_{n-3} - a_{n-4}$, $a_0 = 0$, $a_1 = 1$, $a_2 = 8$, $a_3 = 27$

(i) $a_4 = 4(27) - 6(8) + 4(1) = 64$

$a_5 = 4(64) - 6(27) + 4(8) - 1 = 125$

$a_6 = 4(125) - 6(64) + 4(27) - 8 = 216$

$a_7 = 4(216) - 6(125) + 4(64) - 27 = 343$

$a_8 = 4(343) - 6(216) + 4(125) - 64 = 512$

(ii) $a_n = n^3$

(iii) $a_0 = (0)^3 = 0$

$a_1 = (1)^3 = 1$

$a_2 = (2)^3 = 8$

$a_3 = (3)^3 = 27$

$a_4 = (4)^3 = 64$

$n^3 = 4(n-1)^3 - 6(n-2)^3 + 4(n-3)^3 - (n-4)^3$

$n^3 = 4(n-1)(n^2-2n+1) - 6(n-2)(n^2-4n+4) + 4(n^2-6n+9)(n-3) - (n-4)(n^2-8n+16)$

$n^3 = 4(n^3-2n^2+n-n^2+2n-1) - 6(n^3-4n^2+4n-2n^2+8n-8) + 4(n^3-3n^2-6n^2+18n+9n-27) - (n^3-8n^2+16n-4n^2+32n-64)$

$n^3 = 4n^3 - 12n^2 + 12n - 4 - 6n^3 + 36n^2 - 72n + 48 + 4n^3 - 36n^2 + 108n - 108 - n^3 + 12n^2 - 48n + 64$

$n^3 = n^3$ ✓

2. $\dfrac{dy}{dt} = \dfrac{y^3}{(t+1)}$, $y(0) = 1$

$$\int \frac{dy}{y^3} = \int \frac{dt}{(t+1)}$$

$$-\frac{1}{2y^2} = \ln|t+1| + C$$

$$-\frac{1}{2(1)^2} = \ln|1| + C$$

$$-\frac{1}{2} = C$$

$$\cancel{-\frac{1}{2}y^2 = \ln|t+1| - \frac{1}{2}}$$

$$\cancel{y^2 = -2\ln|t+1| + 1}$$

$$\cancel{y^2 = \ln\left(\frac{1}{(t+1)^2}\right) + 1}$$

$$\cancel{\boxed{y = \pm\sqrt{\ln\left(\frac{1}{(t+1)^2}\right) + 1}}}$$

$$-\frac{1}{2y^2} = \ln|t+1| - \frac{1}{2}$$

$$\frac{1}{y^2} = -2\ln|t+1| + 1$$

$$y^2 = \frac{1}{1-2\ln|t+1|}$$

$$\boxed{y = \frac{1}{\sqrt{1-2\ln|t+1|}}}$$

3.

$$y''(t) - 3y'(t) + 2y(t) = 0 \qquad y(0) = 2, \quad y'(0) = 3$$

$$r^2 - 3r + 2 = 0 \qquad r = \frac{3 \pm \sqrt{9 - 4(2)}}{2}$$

$$(r-1)(r-2) = 0$$

$$r_{1,2} = 1, 2 \qquad r = \frac{3 \pm \sqrt{1}}{2} = \frac{3}{2} \pm \frac{1}{2} = 1, 2$$

$$y_1(t) = e^{r_1 t} \qquad y_2(t) = e^{r_2 t}$$

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$$y(t) = C_1 e^{t} + C_2 e^{2t}$$

$$y'(t) = C_1 e^{t} + 2C_2 e^{2t}$$

① $2 = y(0) = C_1 + C_2$

② $3 = y'(0) = C_1 + 2C_2$

① - ② $-1 = -C_2$

$C_2 = 1$

$2 = C_1 + 1$

$C_1 = 1$

$$\boxed{y(t) = e^{t} + e^{2t}}$$

4.

$$\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3-r & -4 \\ 4 & 3-r \end{bmatrix}$$

$$\det \begin{bmatrix} 3-r & -4 \\ 4 & 3-r \end{bmatrix} = (3-r)^2 - (-4)(4) = 0$$

$$9 - 6r + r^2 + 16 = 0$$

$$r^2 - 6r + 25 = 0$$

$$(r^2 - 6r + 9) - 9 + 25 = 0$$

$$\sqrt{(r-3)^2} = \sqrt{-16}$$

$$r - 3 = \pm 4i$$

$$r = 3 \pm 4i$$

$$\begin{bmatrix} 3-(3+4i) & -4 \\ 4 & 3-(3+4i) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 3-(3-4i) & -4 \\ 4 & 3-(3-4i) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 4i & -4 \\ 4 & 4i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4ix - 4y = 0 \qquad y = -ix \qquad\qquad 4ix - 4y = 0 \qquad y = ix$

$4x - 4iy = 0 \qquad x = iy \qquad\qquad 4x + 4iy = 0 \qquad x = -iy$

$$\boxed{\text{For } \lambda_1 = 3+4i, \; v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \quad \text{For } \lambda_2 = 3-4i, \; v_2 = \begin{bmatrix} -i \\ +1 \end{bmatrix}}$$